Journal of Global Scientific Research in Applied Mathematics and Statistics
Vol.10, Issue 9/ pp. 4007-4018/ September 2025

Contents lists available at www.gsjpublications.com

Journal of Global Scientific Research
in Applied Mathematics and Statistics

journal homepage: www.gsjpublications.com/jourgsr.html

Study of Strong Transitive Random Dynamical Systems

Salam Tawfiq Abbas[1], Qays Arif Wali[2], Walaa Naser Khudhair[3]

[1]The General Directorate for Education of Diyala, Diyala, Iraq.
[2]The General Directorate for Education of Muthanna, Muthanna, Iraq.
[3]The General Directorate for Education of Al-Qadisiyyah, Al-Qadisiyyah, Iraq.

A R T I C L E I N F O

Received: 17 Jul 2025,
Revised: 23 Jul 2025,
Accepted: 3 Aug 2025,
Online: 17 Sep 2025

Keywords:
Random dynamical systems, strong transitivity, strongly dense, Trajectories, strong transitive point, nowhere strong dense.

A B S T R A C T

In this paper establishes fundamental results on strong transitivity in random dynamical systems (RDS). We prove two key equivalences:
Strong transitivity ⇔ nowhere strong dense
Strong transitivity ⇔ random open set V(ω) is a strongly dense set of X.
And prove the dense pullback trajectories is strong transitivity, strong transitivity of RDS is a dynamical property. We investigate strong transitivity in random dynamical systems (RDS), establishing equivalences with dense sets and omega-limit sets, and analyzing their interrelationships.

1. Introduction

Strong transitivity in random dynamical systems (RDS) represents a fundamental extension of topological transitivity to stochastic frameworks. While classical dynamical systems theory has well-established characterizations of transitivity, the random counterpart requires careful consideration of both topological and measurable structures. This paper establishes three pivotal contributions:

Before studying **strong transitivity** of random dynamical systems, we will give some priorities and definitions of random dynamical systems and start from dynamical systems **[5].** where the a pair $(\mathcal{X},\theta)$ is dynamical systems, where $\mathcal{X}$ is a Hausdorff topological space and $\theta:\mathbb{R}\times\mathcal{X}\to\mathcal{X}$ is a mapping such that

(i) θ is continuous;

(ii) $\theta(0,x)=x$ for all $x\in\mathcal{X}$;

(iii) $\theta(t+s,x)=\theta(t,\theta(s,x))$ $\forall\,,t,s\in\mathbb{R},x\in\mathcal{X}$.

Corresponding author:
E-mail addresses: salamalsaedi7@gmail.com (Salam), qaysarif91@gmail.com (Walaa), walaa.nk94@gmail.com (Qays)
doi: 10.5281/jgsr.2025.17108117

We further recall the definition of **metric dynamical system**[2]where the metric dynamical system is a 5-tuple $(T, \Omega, \mathcal{F}, \mathbb{P}, \theta)$

if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and

(i) $\theta: T \times \Omega \to \Omega$ is $(\mathcal{B}(T) \otimes \mathcal{F}, \mathcal{F})$ −measurable,
(ii) $\theta(0, \omega) = \omega$, $\forall \omega \in \Omega$
(iii) $\theta(t+s, \omega) = \theta(t, \theta(s, \omega))$ $\forall$, $t, s \in T, \omega \in \Omega$
(iv) $\mathbb{P}(\theta_t(F)) = \mathbb{P}(F)$, for each $F \in \mathcal{F}$ and for each $t \in T$.
In short, we will write that ($\theta: T \times \Omega \to \Omega$) to $\theta_t\omega$ or $\theta(t, \omega)$

We also remember the definition of **Random Dynamical System** [3] (Shortly $\mathcal{RDS}$) where a measurable $\mathcal{RDS}$ on the measurable space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ over metric dynamical system $(\mathbb{T}, \Omega, \mathcal{F}, \mathbb{P}, \theta)$, with time is mapping $\varphi: \mathbb{T} \times \Omega \times \mathcal{X} \to \mathcal{X}$, having these properties:

1) Measurability, φ is $\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}, \mathcal{B}$ − measurable.
2) (Cocycle property) : The mappings $\varphi(t, \omega) \coloneqq \varphi(t, \omega, \cdot): \mathcal{X} \to \mathcal{X}$ form a cocycle over $\theta(\cdot)$,
i. e. they satisfy $\varphi(0, \omega, x) = x$ for all $\omega \in \Omega$ (if $0 \in \mathbb{T}$)
$\varphi(t+s, \omega) = \varphi(t, \theta(s)\omega) \circ \varphi(s, \omega)$ for all $s, t \in \mathbb{T}, \omega \in \Omega$

We will write the $\mathcal{RDS}$ is denoted by (θ, φ) rather than $(T, \Omega, \mathcal{X}, \theta, \varphi)$

Definition(1.1):[8] Trajectories

Let $D: \omega \mapsto D(\omega)$ be a measurable multifunction. We define the pullback trajectories emerging from D (from time t onward) as the multifunction: $\omega \longmapsto \gamma_D^t(\omega)$given by:
$\gamma_D^t(\omega) \coloneqq \bigcup_{k \geq t} \varphi(k, \theta_{-k}\omega) \mathrm{D}(\theta_{-k}\omega)$

Special Case (Single-Valued Version):
When $D(\omega) = \{\upsilon(\omega)\}$ is single-valued, we denote: $\omega \longmapsto \gamma_\upsilon(\omega) \equiv \gamma_D^0(\omega) \coloneqq \bigcup_{k \geq 0} \varphi(k, \theta_{-k}\omega) \upsilon(\theta_{-k}\omega)$
and call this the pullback trajectory (or orbit) emanating from υ.

Definition (1.2): [8] omega-limit set

Assume $D: \omega \mapsto D(\omega)$ be a multifunction. The omega-limit set is the multifunction of trajectories emanating from D (pull back),

$$\omega \longmapsto \Gamma_D(\omega) \coloneqq \bigcap_{t > o} \overline{\gamma_D^t(\omega)} = \bigcap_{t > o} \overline{\bigcup_{k \geq t} \varphi(k, \theta_{-k}\omega) \mathrm{D}(\theta_{-k}\omega)}$$

if $D(\omega) = \{x(\omega)\}$,is single valued function

Then $\omega \longmapsto \Gamma_x(\omega)$ is called the (pull back) omega-limit set of trajectory (or orbit) emanating from x.

2. Strong transitivity: The following definition captures strong transitivity for random dynamical systems.

Definition (2.1): A $\mathcal{RDS}$ $(\theta, \varphi) on\ \mathcal{X}$ is said to be a strong transitivity, if for each non-empty random open sets $\mathcal{U}(\omega)\ and\ \mathcal{V}(\omega)$ of $\mathcal{X}$, there is $k_1, k_2 \in \mathbb{N}$ such that

$\varphi(k_1, \theta_{-k_1}\omega)\mathcal{U}(\omega) \cap \varphi(-k_2, \omega)\mathcal{V}(\omega) \neq \emptyset$.

The following definition formalizes a strengthened notion of density for random subsets, ensuring non-empty intersection with all random open sets in the state space.

Definition (2.1): Strongly dense set

A random sub set $\varphi(-k,\omega)\mathcal{V}(\omega)$ of $\mathcal{X}$ is said to be a strongly dense set of $\mathcal{X}$, if $\varphi(-k,\omega)\mathcal{V}(\omega)\cap\mathcal{U}(\omega)\neq\emptyset$ for each non-empty random open set $\mathcal{U}(\omega)$ of $\mathcal{X}$.

Main results : Strong transitivity systems have strongly dense.

Theorem (2.2) : if (θ,φ) is strong transitivity, then $\varphi(-k_2,\omega)\mathcal{V}(\omega)$ is a strongly dense set of $\mathcal{X}$ for each non-empty random open set $\mathcal{U}(\omega)\subset\mathcal{X}$.

proof : Let (θ,φ) is strong transitivity, then $\varphi(k_1,\theta_{-k_1}\omega)\mathcal{U}(\omega)\cap\varphi(-k_2,\omega)\mathcal{V}(\omega)\neq\emptyset$ for each non-empty random open sets $\mathcal{U}(\omega)$ *and* $\mathcal{V}(\omega)$ of $\mathcal{X}$, there is $k_1,k_2\in\mathbb{N}$,

since $\mathcal{U}(\omega)\subset\varphi(k_1,\theta_{-k_1}\omega)\mathcal{U}(\omega)$,so $\mathcal{U}(\omega)\cap\varphi(-k_2,\omega)\mathcal{V}(\omega)\neq\emptyset$,there fore $\varphi(-k_2,\omega)\mathcal{V}(\omega)$ is strongly dense set of $\mathcal{X}$.

Theorem (2.3):

Let (θ,φ) be a $\mathcal{RDS}$. The following statements are equivalent:

1) (θ,φ) is a strong transitivity.
2) For each non-empty random open set $\mathcal{V}(\omega)$ of $\mathcal{X}$, $\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{V}(\omega)\}$ is strongly dense set of $\mathcal{X}$.

Proof:

(1) ⇒ (2) Assume that (θ,φ) be a strong transitivity $\mathcal{RDS}$. From transitivity of (θ,φ), implies that for every random open sets $\mathcal{U}(\omega)$ and $\mathcal{V}(\omega)$ of $\mathcal{X}$ there are $k_1,k_2\in\mathbb{N}$ such that:

$$\varphi(k_1,\theta_{-k_1}\omega)\mathcal{U}(\omega)\cap\varphi(-k_2,\omega)\mathcal{V}(\omega)\neq\emptyset$$

$$\varphi(-k_1,\omega)\circ\varphi(k_1,\theta_{-k_1}\omega)\mathcal{U}(\omega)\cap\varphi(-k_1,\omega)\circ\varphi(-k_2,\omega)\mathcal{V}(\omega)\neq\emptyset$$

$$\varphi(k_1+-k_1,\theta_{-k_1}\omega)\mathcal{U}(\omega)\cap\varphi(-k_1,\omega)\circ\varphi(-k_2+-k_1,\omega)\mathcal{V}(\omega)\neq\emptyset$$

Let $-k_2+-k_1=-k_3$

$$\mathcal{U}(\omega)\cap\ \varphi(-k_3,\omega)\mathcal{V}(\omega)\neq\emptyset$$

$$\bigcup_{k_3\in\mathbb{N}}\{\varphi(-k_3,\omega)\mathcal{V}(\omega)\}\cap\ \mathcal{U}(\omega)\ \neq\ \emptyset$$

Then from definition strongly dense set we have

$\bigcup_{k_3\in\mathbb{N}}\{\varphi(-k_3,\omega)\mathcal{V}(\omega)\}$ is strong dense set of $\mathcal{X}$.

(2) ⟹ (1)

Let $\mathcal{V}(\omega)\subset\mathcal{X}$ *be an* random open *set we have* ,

$\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{V}(\omega)\}$ is strongly dense set of $\mathcal{X}$

Let $\mathcal{U}(\omega)$ $be\ any$ random open $set\ subset$ of $\mathcal{X}$

Then there exists $k_1 \in \mathbb{N}$ such that $\varphi(k_1, \theta_{-k_1}\omega)\mathcal{U}(\omega)$ is random open set in $\mathcal{X}$ from definition strongly dense set we have

$\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{V}(\omega)\} \cap \varphi(k_1, \theta_{-k_1}\omega)\mathcal{U}(\omega) \neq \emptyset$

Then there exists $k_1, k_2 \in \mathbb{N}$ such that

$\varphi(k_1, \theta_{-k_1}\omega)\mathcal{U}(\omega) \cap \varphi(-k_2,\omega)\mathcal{V}(\omega) \neq \emptyset$

Hence (θ,φ) is a strong transitivity.

In the following theorem, we will prove that (if $\mathcal{X} = \Gamma_x(\omega)$ (omega-limit set) then (θ,φ) is a strong transitivity)

Theorem (2.4):

Let (θ,φ) be a $\mathcal{RDS}$ if $\mathcal{X} = \Gamma_x(\omega)$ for every $x(\omega) \in \mathcal{X}$ then (θ,φ) is a strong transitivity.

Proof:

assume that $\mathcal{X} = \Gamma_x(\omega)$ and for every random open set as $\varphi(k_1, \theta_{-k_1}\omega)\mathcal{U}(\omega)$ of $\mathcal{X}$ satisfy $\varphi(k_1, \theta_{-k_1}\omega)\mathcal{U}(\omega) \cap \Gamma_x(\omega) \neq \emptyset$

there is $x(\omega) \in \{\varphi(k_1, \theta_{-k_1}\omega)\mathcal{U}(\omega) \cap \Gamma_x(\omega)\}$

we get $x(\omega) \in \varphi(k_1, \theta_{-k_1}\omega)\mathcal{U}(\omega)$ … (1)

and let $\mathcal{V}(\omega)$ is random open set of $\mathcal{X}$ then $\mathcal{V}(\omega) \cap \Gamma_x(\omega) \neq \emptyset$, for some $x(\omega) \in \mathcal{X}$.

That is, there exists $k_2 \in \mathbb{N}$ such that $y = \varphi(k_2, \theta_{-k_2}\omega)x(\omega) \in \mathcal{V}(\omega)$.

Thus, $\varphi(k_2,\omega)^{-1} \circ \varphi(k_2, \theta_{-k_2}\omega)x \in \varphi(k_2,\omega)^{-1}\mathcal{V}(\omega)$.

$x(\omega) \in \varphi(-k_2,\omega)\mathcal{V}(\omega)$ … (2)

From (1) and (2), we have $\varphi(k_1, \theta_{-k_1}\omega)\mathcal{U}(\omega) \cap \varphi(-k_2,\omega)\mathcal{V}(\omega) \neq \emptyset$

hence (θ,φ) is a strong transitivity.

Theorem (2.5):

Let (θ,φ) be $\mathcal{RDS}$ on $\mathcal{X}$, if for each a non-empty random open $\mathcal{U}(\omega)$ subset of $\mathcal{X}$, there is $k \in \mathbb{N}$ such that $\bigcup_{k\geq t}\varphi(k, \theta_{-k}\omega)\,\mathcal{U}(\omega) = \mathcal{X}$,Then (θ,φ) is a strong transitivity.

Proof:

Let $\mathcal{U}(\omega)$ be non-empty random open subset of $\mathcal{X}$

Then there is $k \in \mathbb{N}$ such that $\bigcup_{k\geq t}\varphi(k, \theta_{-k}\omega)\,\mathcal{U}(\omega) = \mathcal{X}$

let $\varphi(-k_2,\omega)\mathcal{V}(\omega)$ subset of $\mathcal{X}$ then there is $\mathcal{Z}(\omega) \in \mathcal{U}(\omega)$ such that

$\varphi(k,\theta_{-k}\omega)\,\mathcal{Z}(\omega) \in \varphi(-k_2,\omega)\,\mathcal{V}(\omega)$ for some k

Then $\varphi(k_1,\theta_{-k_1}\omega)\mathcal{U}(\omega) \cap \varphi(-k_2,\omega)\mathcal{V}(\omega) \neq \emptyset$ for some$k_1 and\ k_2 \in \mathbb{N}$

$\Rightarrow (\theta,\varphi)$ is a strong transitivity.

Theorem (2.6):

Strong Transitivity of $\mathcal{RDS}$ is a dynamical property.

Proof:

Let (θ,φ) and (θ,ψ) be two $\mathcal{RDS}'$s on $\mathcal{X}$ and $\mathcal{Y}$ respectively such that (θ,φ)equivalence(θ,ψ).

Then there exists a homeomorphism mapping $g : (\theta,\varphi) \longrightarrow (\theta,\psi)$. We must prove that if (θ,φ) is a strong transitivity, then (θ,ψ) is a strong transitivity.

Let $\mathcal{G}(\omega)$ and $\mathcal{H}(\omega)$ be two open random sets of $\mathcal{Y}$. From the continuity of g, we have $g^{-1}(\mathcal{G}(\omega))$ and $g^{-1}(\mathcal{H}(\omega))$ are two random open sets of $\mathcal{X}$.

Put $\mathcal{U}(\omega) = g^{-1}(\mathcal{G}(\omega))$ and $\mathcal{V}(\omega) = g^{-1}(\mathcal{H}(\omega))$,

and by using definition (2.1), there is $k \in \mathbb{N}$ such that

$\varphi(k_1,\theta_{-k_1}\omega)\mathcal{U}(\omega) \cap \varphi(-k_2,\omega)\mathcal{V}(\omega) \neq \emptyset$.

It follows that $g(\ \varphi(k_1,\theta_{-k_1}\omega)\mathcal{U}(\omega) \cap \varphi(-k_2,\omega)\mathcal{V}(\omega)) \neq \emptyset$. Then,

$(\psi(k_1,\theta_{-k_1}\omega)\mathcal{U}(\omega) \cap \psi(-k_2,\omega)\mathcal{V}(\omega)) \ \neq \emptyset$.

By using definition (2.1) again, we have (θ,ψ) is strong transitivity.

Theorem (2.7) :

If two $\mathcal{RDS}$ (θ_1,φ_1) , (θ_2,φ_2) are strong transitive's, then their product is a strong transitivity.

Proof:

Let (θ_1,φ_1) , (θ_2,φ_2) are two strong transitive's we must prove $(\theta_1,\varphi_1) \times (\theta_2,\varphi_2)$ is a strong transitive, now define $(\theta,\varphi) = (\theta_1,\varphi_1) \times (\theta_2,\varphi_2)$.

Define $\mathcal{U}(\omega_1,\omega_2) = \mathcal{U}_1(\omega_1) \times \mathcal{U}_2(\omega_2)$ where $\omega_1,\omega_2 \in \Omega$,

$\varphi(-k_2,\omega)\mathcal{V}(\omega_1,\omega_2) = \varphi(-k_2,\omega)\mathcal{V}_1(\omega_1) \times \varphi(-k_2,\omega)\mathcal{V}_2(\omega_2)$ where $\omega_1,\omega_2 \in \Omega$,

$\varphi(k_1,\theta_{-k_1}\omega)\mathcal{U}(\omega_1,\omega_2) = \varphi\left(k_1,\theta_{-k_1}\omega\right)\mathcal{U}_1(\omega_1) \times \varphi\left(k_2,\theta_{-k_2}\omega\right)\mathcal{U}_2(\omega_2)$

Where $k_1,k_2, \in \mathbb{N}$.

We must prove $\varphi(k_1,\theta_{-k_1}\omega)\mathcal{U}(\omega_1,\omega_2) \cap \varphi(-k_2,\omega)\,\mathcal{V}(\omega_1,\omega_2) \neq \emptyset$.

Since $\mathcal{U}_1(\omega_1), \mathcal{U}_2(\omega_2)$ are two random open sets, then $\mathcal{U}(\omega_1, \omega_2)$ is a random open set also $\varphi(-k_2, \omega)\mathcal{V}_1(\omega_1), \varphi(-k_2, \omega)\mathcal{V}_2(\omega_2)$ are two random open sets, then $\varphi(-k_2, \omega)\mathcal{V}(\omega_1, \omega_2)$ is a random open set. since (θ_1, φ_1) is a strong transitivity, then $\varphi(k_1, \theta_{-k_1}\omega)\mathcal{U}_1(\omega_1) \cap \varphi(-k_2, \omega)\mathcal{V}_1(\omega_1) \neq \emptyset$. Also, since (θ_2, φ_2) is a strong transitivity, then $\varphi(k_2, \theta_{-k_2}\omega)\mathcal{U}_2(\omega_2) \cap \varphi(-k_2, \omega)\mathcal{V}_2(\omega_2) \neq \emptyset$, we get:

$$\varphi(k, \theta_{-k}\omega)\mathcal{U}(\omega_1, \omega_2) \cap \varphi(-k_2, \omega)\mathcal{V}(\omega_1, \omega_2)$$

$$= \{\varphi(k_1, \theta_{-k_1}\omega)\mathcal{U}_1(\omega_1) \times \varphi(k_2, \theta_{-k_2}\omega)\mathcal{U}_2(\omega_2)\} \cap \{\varphi(-k_2, \omega)\mathcal{V}_1(\omega_1) \times \varphi(-k_2, \omega)\mathcal{V}_2(\omega_2)\}$$

$$= \{\varphi(k_1, \theta_{-k_1}\omega)\mathcal{U}_1(\omega_1) \times \varphi(k_2, \theta_{-k_2}\omega)\mathcal{U}_2(\omega_2)\} \wedge \{\varphi(-k_2, \omega)\mathcal{V}_1(\omega_1) \times \varphi(-k_2, \omega)\mathcal{V}_2(\omega_2)\}$$

$$= \{\varphi(k_1, \theta_{-k_1}\omega)\mathcal{U}_1(\omega_1) \wedge \varphi(-k_2, \omega)\mathcal{V}_1(\omega_1)\} \times \{\varphi(k_2, \theta_{-k_2}\omega)\mathcal{U}_2(\omega_2) \wedge \varphi(-k_2, \omega)\mathcal{V}_2(\omega_2)\}$$

$$= \varphi(k_1, \theta_{-k_1}\omega)\mathcal{U}_1(\omega_1) \cap \varphi(-k_2, \omega)\mathcal{V}_1(\omega_1) \times \varphi(k_2, \theta_{-k_2}\omega)\mathcal{U}_2(\omega_2) \cap \varphi(-k_2, \omega)\mathcal{V}_2(\omega_2) \neq \emptyset.$$

Then $\varphi(k, \theta_{-k}\omega)\mathcal{U}(\omega_1, \omega_2) \cap \varphi(-k_2, \omega)\mathcal{V}(\omega_1, \omega_2) \neq \emptyset$ and hence (θ, φ) is a strong transitivity.

3. strong transitivity point

In the context of random dynamical systems, strong transitivity points play a fundamental role in understanding the ergodic and transitive behavior of trajectories. The following definition precisely specifies the conditions required for a random variable to be a strong transitivity point of the dynamical system (θ, φ).

Definition (3.1): strong transitivity point

A random variable $x(\omega) \in \mathcal{X}_{\mathcal{B}}^{\Omega}$ is said to be a strong transitivity point to (θ, φ) if its orbit $\varphi(k, \omega)\, x(\theta_k\omega)$ is a strongly dense in $\mathcal{X}$.

Theorem (3.2): if some point $x \in \mathcal{X}_{\mathcal{B}}^{\Omega}$ is a strong transitivity point, then (θ, φ) is a strong transitivity.

Proof:

If $x(\omega) \in \mathcal{X}_{\mathcal{B}}^{\Omega}$ is a strong transitivity point and if $\mathcal{U}(\omega)$ *and* $\mathcal{V}(\omega)$ are non-empty random open sets of $\mathcal{X}$, there exist integers $k_1, k_2 \in \mathbb{N}$ with

$$\varphi(k_1, \omega)\, x(\theta_{k_1}\omega) \in \mathcal{U}(\omega)$$

$$x(\theta_{k_1}\omega) \in \varphi(k_1, \theta_{-k_1}\omega)\mathcal{U}(\omega) \ldots (1)$$

and $\varphi(k_2, \omega)\, x(\theta_{k_2}\omega) \in \mathcal{V}(\omega)$

$$\varphi(-k_2, \omega) \circ \varphi(k_2, \omega)\, x(\theta_{k_2}\omega) \in \varphi k_2(-k_2, \omega) \circ \mathcal{V}(\omega)$$

$$\varphi(-k_2 + k_2, \omega)\, x(\theta_{k_2}\omega) \in \varphi(-k_2, \omega)\, \mathcal{V}(\omega)$$

$$\varphi(0, \omega)\, x(\theta_{k_2}\omega) \in \varphi(-k_2, \omega)\, \mathcal{V}(\omega)$$

$$x(\theta_{k_2}\omega) \in \varphi(-k_2, \omega)\, \mathcal{V}(\omega) \ldots (2)$$

from (1) and (2) we get $\varphi(k_1, \theta_{-k_1}\omega)\mathcal{U}(\omega) \cap \varphi(-k_2, \omega)\, \mathcal{V}(\omega) \neq \emptyset$

,hence (θ, φ) is a strong transitivity.

Definition (3.3): nowhere strongly dense

$\mathcal{X}$ is said to be nowhere strong dense if it has an empty.

Theorem (3.4): For any $\mathcal{RDS}(\theta, \varphi)$ the following statements are equivalent :

1) (θ, φ) is a strong transitivity.

2) if $\mathcal{C}(\omega) \subset \mathcal{X}$ is a random closed set and $\varphi(k, \theta_{-k}\omega)\, \mathcal{C}(\omega) \subset \mathcal{C}(\omega)$, then $\mathcal{C}(\omega) = \mathcal{X}$ or $\mathcal{C}(\omega)$ is nowhere strongly dense.

Proof : $(1 \Rightarrow 2)$

Let (θ, φ) is a strong transitivity for each pair of non-empty random open sets $\mathcal{U}(\omega), \mathcal{V}(\omega) \subset \mathcal{X}$ there is an integer $k_1, k_2 \in \mathbb{Z}$ such that

$\varphi\left(k_1, \theta_{-k_1}\omega\right)\mathcal{U}(\omega) \cap \varphi(-k_2, \omega)\, \mathcal{V}(\omega) \neq \emptyset$ and let $\mathcal{C}(\omega) \subset \mathcal{X}$ is a random closed set and $\varphi(k, \theta_{-k}\omega)\, \mathcal{C}(\omega) \subset \mathcal{C}(\omega)$.

Assume $\mathcal{C}(\omega) \neq \mathcal{X}$ and $\mathcal{C}(\omega)$ have a non-empty interior.

Define $\mathcal{U}(\omega) = \mathcal{X} / \mathcal{C}(\omega) \Rightarrow \mathcal{U}(\omega)$ is a random open set (since $\mathcal{C}(\omega)$ is closed)

Let $\mathcal{V}(\omega) \subset \mathcal{C}(\omega)$ be a random open set (since $\mathcal{C}(\omega)$ has a non-empty interior)

We have $\varphi(k, \theta_{-k}\omega)\mathcal{V}(\omega) \subset \mathcal{C}(\omega)$ (since $\varphi(k, \theta_{-k}\omega)\, \mathcal{C}(\omega) \subset \mathcal{C}(\omega)$)

$\varphi\left(k, \theta_{-k_1}\omega\right)\mathcal{V}(\omega) \cap \mathcal{U}(\omega) = \emptyset$ for every k$\in \mathbb{N}$

$\varphi(-k_2, \omega) \circ \varphi\left(k_1, \theta_{-k_1}\omega\right)\mathcal{V}(\omega) \cap \varphi(-k_2, \omega) \circ \mathcal{U}(\omega) = \emptyset$

$\varphi\left(-k_2 + k_1, \theta_{-k_1}\omega\right) \mathcal{V}(\omega) \cap \varphi(-k_2, \omega)\mathcal{U}(\omega) = \emptyset$

Let $-k_2 + k_1 = k_3 \in \mathbb{Z}$

Therefore $\varphi\left(k_3, \theta_{-k_3}\omega\right)\mathcal{V}(\omega) \cap \varphi(-k_2, \omega)\mathcal{U}(\omega) = \emptyset$

This is a contradiction, since (θ, φ) is a strong transitivity, hence $\mathcal{C}(\omega) = \mathcal{X}$ or $\mathcal{C}(\omega)$ is nowhere strongly dense.

$(2 \Longrightarrow 1)$

Let $\mathcal{G}(\omega)$ be a non-empty random open set in $\mathcal{X}$. Suppose (θ, φ) is not strong transitivity for every non-empty random open set in $\mathcal{X}$. Then from theorem (2.3) part (2) we have $\bigcup_{k\in\mathbb{N}}\{\varphi(-k, \omega)\mathcal{U}(\omega)\}$ is not strongly dense, but $\bigcup_{k\in\mathbb{N}}\{\varphi(-k, \omega)\mathcal{U}(\omega)\}$ is random open set. Define $\mathcal{C}(\omega) = \mathcal{X} / \bigcup_{k\in\mathbb{N}}\{\varphi(-k, \omega)\mathcal{U}(\omega)\}$, then $\mathcal{C}(\omega)$ is the random closed set of $\mathcal{X}$ and $\mathcal{C}(\omega) \neq \mathcal{X}$. Claim $\varphi(k, \theta_{-k}\omega)\, \mathcal{C}(\omega) \subset \mathcal{C}(\omega)$. Suppose $\varphi(k, \theta_{-k}\omega)\, \mathcal{C}(\omega)$ is not a subset of $\mathcal{C}(\omega)$, this implies $\varphi(k, \theta_{-k}\omega)\, \mathcal{C}(\omega) \cap \bigcup_{k\in\mathbb{N}}\{\varphi(-k, \omega)\mathcal{U}(\omega)\} \neq \emptyset$.

This further implies:

$$\varphi(-k,\omega)\circ\left[\varphi(k,\theta_{-k}\omega)\mathcal{C}(\omega)\cap\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{U}(\omega)\}\right]$$

$= \mathcal{C}(\omega)\cap\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{U}(\omega)\}\neq\emptyset$, this is a contradiction to definition of $\mathcal{C}(\omega)$, thus $\varphi(k,\theta_{-k}\omega)\,\mathcal{C}(\omega)\subset\mathcal{C}(\omega)$. Since $\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{U}(\omega)\}$ is not strongly dense, there exists a non-empty random open set $\varphi(-k,\omega)\mathcal{H}(\omega)$ in $\mathcal{X}$ such that $\bigcup_{k\in\mathbb{N}}\{\varphi(-k,\omega)\mathcal{U}(\omega)\}\cap\varphi(-k,\omega)\mathcal{H}(\omega)=\emptyset$, this implies $\varphi(-k,\omega)\mathcal{H}(\omega)\subset\mathcal{C}(\omega)$. Then $\varphi(-k,\omega)\mathcal{H}(\omega)\cap\mathcal{C}(\omega)\neq\emptyset$, this is a contradiction to the fact that $\mathcal{C}(\omega)$ is nowhere strongly dense, so (θ,φ) is a strong transitivity.

4. Let us define strongly closed transitive random sets comprising strongly dense subsets and characterize their association with strong transitivity phenomena.

Definition (4.1): (strongly closed transitive random sets)

Let (θ,φ) be $\mathcal{RDS}$ on $\mathcal{X}$. a random closed set $\mathcal{A}$ subset of $\mathcal{X}$ is said to be a strongly transitive set if it contains strongly dense set $\mathcal{V}^{\mathcal{A}}$ subset of $\mathcal{A}$.

Theorem (4.2): If (θ,φ) be $\mathcal{RDS}$ on $\mathcal{X}$ is a strong transitivity, then every random closed subset of $\mathcal{X}$ is a strongly transitive set.

Proof :

Let $\mathcal{A}(\omega)$ be non-empty random closed subset of $\mathcal{X}$ and let $\mathcal{V}^{\mathcal{A}}(\omega)$ be random open subset of $\mathcal{A}(\omega)$

Since (θ,φ) is a strong transitivity then $\varphi(k_1,\theta_{-k_1}\omega)\mathcal{U}(\omega)\cap\varphi(-k_2,\omega)\mathcal{V}^{\mathcal{A}}\mathcal{V}(\omega)\neq\emptyset$ for each non-empty random open sets $\mathcal{U}(\omega)$ *and* $\mathcal{V}^{\mathcal{A}}(\omega)$ of $\mathcal{X}$, there is $k_1,k_2\in\mathbb{N}$

Since $\mathcal{U}(\omega)\subset\varphi(k_1,\theta_{-k_1}\omega)\mathcal{U}(\omega)$

Then $\varphi(-k,\omega)\mathcal{V}^{\mathcal{A}}(\omega)\cap\mathcal{U}(\omega)\neq\emptyset$ this means that $\varphi(-k,\omega)\mathcal{V}^{\mathcal{A}}(\omega)$ is strongly dense set , hence $\mathcal{A}(\omega)$ is a strongly transitive set.

Theorem (4.3): Let $\mathcal{C}(\omega)$*and* $\mathcal{D}(\omega)$ be strongly transitive random closed sets. Then their Product $\mathcal{C}(\omega)\times\mathcal{D}(\omega)$is strongly transitive.

Proof:

Let $\mathcal{C}(\omega),\mathcal{D}(\omega)$ are two strongly closed transitive random sets

since $\mathcal{C}(\omega)$ is strongly transitive random closed set, then there exists strong dense set $\mathcal{V}^{\mathcal{C}}(\omega_1)$ subset of $\mathcal{C}(\omega)$ and random open $\mathcal{U}_1(\omega_1)$ subset of $\mathcal{X}$ such that

$\mathcal{U}_1(\omega_1)\cap\varphi(-k,\omega)\mathcal{V}^{\mathcal{C}}(\omega_1)\neq\emptyset$. where $\omega_1\in\Omega$

Also since $\mathcal{D}(\omega)$ is strongly transitive random closed set, then there exists strong dense set $\mathcal{V}^{\mathcal{D}}(\omega_2)$ subset of $\mathcal{D}(\omega)$ and random open $\mathcal{U}_2(\omega)$ subset of $\mathcal{X}$ such that

$\mathcal{U}_2(\omega_2)\cap\varphi(-k,\omega)\mathcal{V}^{\mathcal{D}}(\omega_2)\neq\emptyset$. where $\omega_2\in\Omega$

Define $\mathcal{A}(\omega)=\mathcal{C}(\omega)\times\mathcal{D}(\omega)$

Let $\mathcal{V}^{\mathcal{A}}(\omega_1,\omega_2)$ be strongly dense set of $\mathcal{A}(\omega)$ and $\mathcal{U}(\omega_1,\omega_2)$ subset of $\mathcal{X}$

Define $\mathcal{U}(\omega_1,\omega_2) = \mathcal{U}_1(\omega_1) \times \mathcal{U}_2(\omega_2)$ where $\omega_1,\omega_2 \in \Omega$,

$\mathcal{V}^{\mathcal{A}}(\omega_1,\omega_2) = \mathcal{V}^{\mathcal{C}}(\omega_1) \times \mathcal{V}^{\mathcal{D}}(\omega_2)$ where $\omega_1,\omega_2 \in \Omega$,

$\varphi(-k,\omega)\mathcal{V}^{\mathcal{A}}(\omega_1,\omega_2) = \varphi(-k_1,\omega)\mathcal{V}^{\mathcal{C}}(\omega_1) \times \varphi(-k_2,\omega)\mathcal{V}^{\mathcal{D}}(\omega_2)$

Where $k_1, k_2, \in \mathbb{N}$

We must prove $\mathcal{U}(\omega_1,\omega_2) \cap \varphi(-k,\omega)\mathcal{V}^{\mathcal{A}}(\omega_1,\omega_2) \neq \emptyset$.

Since $\mathcal{U}_1(\omega_1), \mathcal{U}_2(\omega_2)$ are two random open set then $\mathcal{U}(\omega_1,\omega_2)$ is random open set

Since $\mathcal{V}^{\mathcal{C}}(\omega_1), \mathcal{V}^{\mathcal{D}}(\omega_2)$ are two strongly dense set then $\mathcal{V}^{\mathcal{A}}(\omega_1,\omega_2)$ is a strongly dense set,

Then $\mathcal{U}(\omega_1,\omega_2) \cap \varphi(-k,\omega)\mathcal{V}^{\mathcal{A}}(\omega_1,\omega_2)$

$= \{\mathcal{U}_1(\omega_1) \times \mathcal{U}_2(\omega_2)\} \cap \{\varphi(-k_1,\omega)\mathcal{V}^{\mathcal{C}}(\omega_1) \times \varphi(-k_2,\omega)\mathcal{V}^{\mathcal{D}}(\omega_2)\}$

$= \{\mathcal{U}_1(\omega_1) \times \mathcal{U}_2(\omega_2)\} \wedge \{\varphi(-k_1,\omega)\mathcal{V}^{\mathcal{C}}(\omega_1) \times \varphi(-k_2,\omega)\mathcal{V}^{\mathcal{D}}(\omega_2)\}$

$= \{\mathcal{U}_1(\omega_1) \wedge \varphi(-k_1,\omega)\mathcal{V}^{\mathcal{C}}(\omega_1)\} \times \{\mathcal{U}_2(\omega_2) \wedge \varphi(-k_2,\omega)\mathcal{V}^{\mathcal{D}}(\omega_2)\}$

$= \mathcal{U}_1(\omega_1) \cap \varphi(-k_1,\omega)\mathcal{V}^{\mathcal{C}}(\omega_1) \times \mathcal{U}_2(\omega_2) \cap \varphi(-k_2,\omega)\mathcal{V}^{\mathcal{D}}(\omega_2) \neq \emptyset$

Then $\mathcal{U}(\omega_1,\omega_2) \cap \varphi(-k,\omega)\mathcal{V}^{\mathcal{A}}(\omega_1,\omega_2) \neq \emptyset$

Hence $\mathcal{C}(\omega) \times \mathcal{D}(\omega)$ is a strongly transitive.

Corollary (4.4): If some point $x \in \mathcal{X}_{\mathcal{B}}^{\Omega}$ is a strong transitivity point, then every random closed subset of $\mathcal{X}$ is a strongly transitive set.

Proof: Assume $x(\omega) \in \mathcal{X}_{\mathcal{B}}^{\Omega}$ is a strong transitivity point, then:

1. By Theorem (3.2), the system (θ,φ) is strongly transitive.
2. By Theorem (4.2), this implies that every random closed subset of $\mathcal{X}$ is a strongly transitive set.

Therefore, the conclusion follows.

Corollary (4.5): If some point $x \in \mathcal{X}_{\mathcal{B}}^{\Omega}$ is a strong transitivity point, then $\mathcal{C}(\omega) \subset \mathcal{X}$ is nowhere strongly dense.

Proof: Assume $x(\omega) \in \mathcal{X}_{\mathcal{B}}^{\Omega}$ is a strong transitivity point, then:

1. By Theorem (3.2), the system (θ,φ) is strongly transitive.
2. By Theorem (3.4), this implies that $\mathcal{C}(\omega) \subset \mathcal{X}$ is nowhere strongly dense.

   Therefore, the conclusion follows.

Corollary (4.6): If some point $x \in \mathcal{X}_{\mathcal{B}}^{\Omega}$ is a strong transitivity point, then $\varphi(-k_2,\omega)\mathcal{V}(\omega)$ is a strongly dense set of $\mathcal{X}$ for each non-empty random open set $\mathcal{U}(\omega) \subset \mathcal{X}$.

Proof: Assume $x(\omega) \in \mathcal{X}_{\mathcal{B}}^{\Omega}$ is a strong transitivity point, then:

3. By Theorem (3.2), the system (θ, φ) is strongly transitive.
4. By Theorem (2.2), this implies that $\varphi(-k_2, \omega)\mathcal{V}(\omega)$ is a strongly dense set of $\mathcal{X}$.

   Therefore, the conclusion follows.

Corollary (4.7): Let (θ, φ) be a $\mathcal{RDS}$ if $\mathcal{X} = \Gamma_x(\omega)$ for every $x(\omega) \in \mathcal{X}$, then then $\varphi(-k_2, \omega)\mathcal{V}(\omega)$ is a strongly dense set of $\mathcal{X}$ for each non-empty random open set $\mathcal{U}(\omega) \subset \mathcal{X}$.

Proof: if $\mathcal{X} = \Gamma_x(\omega)$ for every $x(\omega) \in \mathcal{X}$, then:

5. By Theorem (2.4), the system (θ, φ) is strongly transitive.
6. By Theorem (2.2), this implies that $\varphi(-k_2, \omega)\mathcal{V}(\omega)$ is a strongly dense set of $\mathcal{X}$.

   Therefore, the conclusion follows.

Corollary (4.8): Let (θ, φ) be a $\mathcal{RDS}$ if $\mathcal{X} = \Gamma_x(\omega)$ for every $x(\omega) \in \mathcal{X}$, then $\mathcal{C}(\omega) \subset \mathcal{X}$ is nowhere strongly dense.

Proof: : if $\mathcal{X} = \Gamma_x(\omega)$ for every $x(\omega) \in \mathcal{X}$,then:

7. By Theorem (2.4), the system (θ, φ) is strongly transitive.
8. By Theorem (3.4), this implies that $\mathcal{C}(\omega) \subset \mathcal{X}$ is nowhere strongly dense.

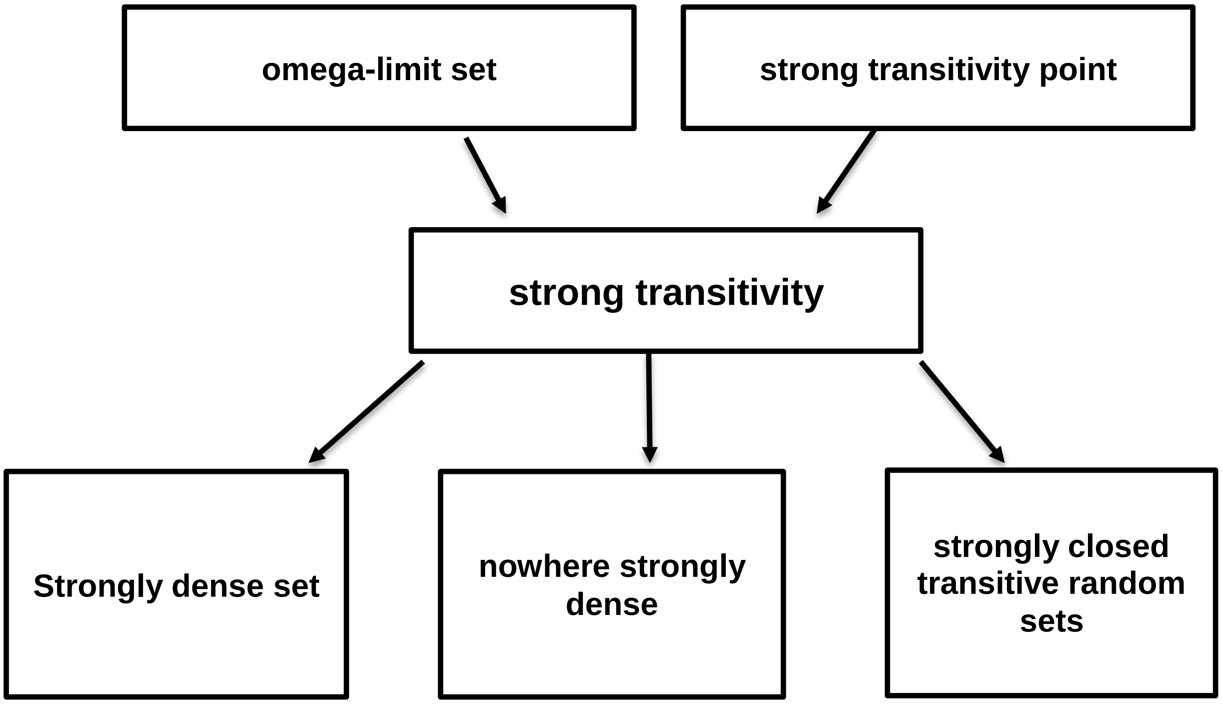
   Therefore, the conclusion follows.

Corollary (4.9): Let (θ, φ) be a $\mathcal{RDS}$ if $\mathcal{X} = \Gamma_x(\omega)$ for every $x(\omega) \in \mathcal{X}$, then every random closed subset of $\mathcal{X}$ is a strongly transitive set.

Proof: : if $\mathcal{X} = \Gamma_x(\omega)$ for every $x(\omega) \in \mathcal{X}$,then:

9. By Theorem (2.4), the system (θ, φ) is strongly transitive.
10. By Theorem (4.2), this implies that every random closed subset of $\mathcal{X}$ is a strongly transitive set.

Therefore, the conclusion follows.



5. Conclusion

This work has systematically characterized strong transitivity in random dynamical systems, unifying topological and measurable perspectives. The key outcomes reveal that:

- Strong transitivity fundamentally links to the density of system trajectories and the minimality of asymptotic behavior, as captured through omega-limit sets.

- The property remains invariant under standard dynamical operations, including system equivalences and products, underscoring its robustness.

- Nowhere-density of certain invariant sets emerges as a critical indicator of transitive behavior.

6. Practical Implications:

These findings enable new approaches to:

1. Modeling ergodicity in stochastic climate systems

2. Analyzing stability in financial markets with random shocks

3. Designing control strategies for noise-driven biological networks

7. Future Directions:

While the theoretical framework is now established, open challenges include:

- Extending results to non-compact state spaces

- Developing numerical verification tools

- Exploring connections to entropy in noisy systems

This research solidifies strong transitivity as a central concept for classifying stochastic dynamics, with potential to bridge theoretical mathematics and applied sciences.

8. Acknowledgments

We extend our sincere gratitude to the Faculty of Science and its libraries for their invaluable support in completing and publishing this research. We deeply appreciate the administration and staff for providing the necessary resources and an optimal research environment that contributed to the success of this work.

9. Conflicts of Interest

The authors declare no conflict of interest.

References

[1]. E. Akin, Jeffrey D. Carlson "Conceptions of topological transitivity" Mathematics Department, The City College, 137 Street and Convent Avenue, New York, NY 10031, USA, Topology and its Applications 159 , (2012).
[2]. L. Arnold "Random Dynamical Systems", Springer-Verlag Berlin Heidelberg Germany (1998).
[3]. R.B. Ash," Probability and Measure Theory", Academic Press, Inc. New York .(2000).
[4]. J. Banks , B. Stanley , "A Note On Equivalent Definitions Of Topological Transitivity " Department of Mathematics and Statistics La Trobe University Plenty Road Bundoora 3086, Victoria, Australia , (2014) .
[5]. L. Barreira, C. Valls, " Dynamical Systems", Universitext, , Springer-Verlag London(2013) .
[6]. A. D. Barwell "ω-Limit Sets of Discrete Dynamical Systems" School of Mathematics College of Engineering and Physical Sciences The University of Birmingham August ,(2010) .
[7]. A.Barzanouni and E. Shah "Chain Transitivity for Maps on G-Spaces" Matematicki Vesnik Matematiqki Vesnik 71,(2019).
[8]. I. Chueshov " Monotone Random Systems Theory and Applications" Springer- verlag Berlin Heidelberg Germany (2002).
[9]. R. Das and M. Garg "Average Chain Transitivity and The AL Most Average Shadowing Property" Commun. Korean Math. (2016).
[10]. R. Das and T. Das "Topological Transitivity of Uniform Limit Functions on G-spaces" Department of Mathematics, Faculty of Science The M. S. University of Baroda, Vadodara-390002, India , Int. Journal of Math. Analysis, Vol. 6, (2012),
[11]. L. Gongfu, W. Lidong & Z. Yucheng , "Transitivity, mixing and chaos for a class of set-valued mappings" Science in China : Series A Mathematics (2006) .
[12]. A. Kameyama " Topological Transitivity and Strong Transitivity" Acta Math. Univ. Comenianae ,(2002) .
[13]. S. Kolyada , L. Snoha , "Some aspects of topological transitivity - A survey" Institute of Mathematics, Ukrainian Academy of Sciences Tereschenkivs'ka 3, 252601 Kiev, Ukraine (1997) .
[14]. P. Raith and A. Stachelberger , "Topological transitivity for a class of monotonic mod one transformations" Aequat. Math. 82 (2011),
[15]. Saleh, A. J. Numerical Solutions of Systems of Nonlinear Volterra Integro-Differential Equations [VIDEs] Using Block Method. *Journal of Global Scientific Research in Applied Mathematics and Statistics*. 7(7); 2452-2463. (2022).