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Some Remarks on One-way Repeated Measurements Model for Unbalanced Data

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ABSTRACT

In this paper, we consider the one-way repeated measurements model for unbalanced data (where the number of observations is unequal in each levels). We then discuss method to estimate the parameters of the model ,and introduce some properties of estimators. As practical research , the objective of the experiment is to study the effect of the experimental factors and their interactions on the production of date palm, cultivar El Halawi. The experiment included the factor of irrigation method with three treatments: drip irrigation, basin irrigation, and the regular tidal method, while the irrigation interval factor included three treatments: 3, 6 and 9 days.

1. Introduction

Repeated measurements analysis is commonly used in several fields, for example biomedical research, epidemiology, health and life sciences, etc. Many literatures were given to the univariate repeated measurements analysis of variance (RMANOVA) [8][11]. The data in which the response variable for each experimental unit is observed on multiple occasions and possibly under different-experimental-conditions term is called repeated measurements [8]**.** Repeated measures designs involving two or more independent groups are among the most common experimental designs in a variety of research settings. Various statistical procedures have been suggested for analyzing data from split-plot designs when parametric model assumptions are violated [9].

These responses could be realized as distinct variables, or repeated measurements of one variable, or repeated measurements of a set of variables. The terminology used for the various RM design is as follows: A one – way RM ANOVA refers to the situation with one within-units factor and a multi – way RM MANOVA to the situation with more than one within-units factor . In RM ANOVA, there are two kinds of data: balanced and unbalanced data. A balance RM design means that the p occasions of the measurement are the same for all of the experimental units while the unbalance data means that the occasions of measurement are not the same for all of the experimental units [10].The repeated measurements model(**RMM)** has been investigated by many researchers. Al-Mouel. in (2004)[3], studied the multivariate repeated measures models and comparison of estimators. AL-Mouel and et. in

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(2017)[1] , studied the Bayesian estimation of one-way repeated measurements model. Bayesian approach according to Bayes quadratic unbiased estimator of the linear one- way repeated measurements model. AL-Mouel and Al-Isawi in (2019)[2], studied best quadratic unbiased estimator of variance components for balanced data for repeated measurement model (RMM), they computed the quadratic unbiased estimator, which has minimum variance (best quadratic unbiased estimate (BQUE)) by using analysis of variance (ANOVA) method of estimating the variance components. AL-Mouel and AL-Hasan in(2021)[5], they studied statistical inference in variance components repeated measurements models. AL-Mouel and Ali in(2021)[6],they studied of random effects in repeated measurements model. AL-Mouel and Abd-Ali in(2021)[4],they studied a variance components estimation for repeated measurements model .

In this paper, the parameters of the model have been evaluated by using ordinary least squares method and take practical side about effect irrigation interval on moisture and salt distribution of data palm orchards soil under different irrigation systems.

2. One-Way Repeated Measurements Model

We consider the one-way repeated measurements model for unbalanced data(where the number of observations is unequal in each levels).A mathematical expression for the model is presented and probability distributions for its components are discussed. .

2.1 The Model

Consider the one-way repeated measurements model for unbalanced data,

$$y_{ijk} = \mu + \alpha_j + \beta_k + (\alpha\beta)_{jk} + \delta_{i(j)} + e_{ijk} , \qquad (2.1.1)$$

where

y_{ijk} is the response measurement at time k for unit i within group j,

$i=1,2,\ldots,n_j$ is an index for experimental unit within group j ,

$j=1,2,\ldots,q$ is an index for levels of the between-units factor (Group) ,

$k=1,2,\ldots,p$ is an index for levels of the within-units factor (Time) ,

μ is the overall mean ,

α_j is the added effect for treatment group j ,

β_k is the added effect for time k ,

$(\alpha\beta)_{jk}$ is the added effect for the group j × time k (Interaction) ,

$\delta_{i(j)}$ is the random effect due to experimental unit i within treatment group j ,

e_{ijk} is the random error time k for unit i within group j .

under the following considerations for the added parameter (added effect)

$$\left.\begin{array}{l}\sum_{j=1}^{q}\alpha_j=0\ ,\sum_{k=1}^{p}\beta_k=0\ ,\quad \sum_{j=1}^{q}(\alpha\beta)_{jk}=0\ ,\forall\, k=1,\dots,p\ ,\\ \sum_{k=1}^{p}(\alpha\beta)_{jk}=0\ ,\quad \forall j=1,\dots,q\ ,\quad with\ \ n=\sum_{j=1}^{q}n_j\ ,and\ \sum_{j=1}^{q}n_j\alpha_j=0\ .\end{array}\right\}\qquad (2.1.2)$$

we suppose that the $\delta_{i(j)}$'s and e_{ijk}'s are independent and identically with

$$\left.\begin{array}{l}e_{ijk}\ \sim i.i.d.N(0,\sigma_e^2)\ ,\\ \delta_{i(j)}\ \sim i.i.d.N\left(0,\sigma_\delta^2\right),\end{array}\right\}\qquad (2.1.3)$$

2.2 Estimation

In this section, we consider the estimators for the parameters by using Ordinary Least Squares Method.

The technique of this method to estimate the model's parameters ,first we define a function Q

$$Q=\sum_{j=1}^{q}\sum_{i=1}^{n_j}\sum_{k=1}^{p}e_{ijk}^2$$

$$Q=\sum_{j=1}^{q}\sum_{i=1}^{n_j}\sum_{k=1}^{p}(y_{ijk}-E\left(y_{ijk}\right))^2,$$

$$Q=\sum_{j=1}^{q}\sum_{i=1}^{n_j}\sum_{k=1}^{p}(y_{ijk}-\mu-\alpha_j-\beta_k-(\alpha\beta)_{jk})^2.\qquad (2.2.1)$$

Then differentiate (2.2.1)with respect to μ and equate the derivative to zero given us

$$\frac{\partial Q}{\partial\mu}=-2\sum_{j=1}^{q}\sum_{i=1}^{n_j}\sum_{k=1}^{p}\left(y_{ijk}-\mu-\alpha_j-\beta_k-(\alpha\beta)_{jk}\right)=0,$$

$$\sum_{j=1}^{q}\sum_{i=1}^{n_j}\sum_{k=1}^{p}y_{ijk}-np\hat{\mu}-p\sum_{j=1}^{q}n_j\alpha_j-n\sum_{k=1}^{p}\beta_k-\sum_{j=1}^{q}n_j\sum_{k=1}^{p}(\alpha\beta)_{jk}=0,$$

From equation (2.1.2), we have

$$\sum_{j=1}^{q}\sum_{i=1}^{n_j}\sum_{k=1}^{p}y_{ijk}-np\hat{\mu}=0,$$

$$\hat{\mu}=\frac{1}{np}\sum_{j=1}^{q}\sum_{i=1}^{n_j}\sum_{k=1}^{p}y_{ijk}$$

$$\therefore\ \hat{\mu}=\bar{y}_{\dots}\ ,\qquad (2.2.2)$$

Now differentiate (2.2.1)with respect to α_j and equate the derivative to zero given us

$$\frac{\partial Q}{\partial \alpha_j} = -2q\sum_{i=1}^{n_j}\sum_{k=1}^{p}\left(y_{ijk} - \mu - \alpha_j - \beta_k - (\alpha\beta)_{jk}\right) = 0\,, \forall j = 1, \dots, q\,,$$

$$= \sum_{i=1}^{n_j}\sum_{k=1}^{p} y_{ijk} - n_j p\hat{\mu} - n_j p\hat{\alpha}_j - n_j\sum_{k=1}^{p}\beta_k - n_j\sum_{k=1}^{p}(\alpha\beta)_{jk} = 0\,, \forall j = 1, \dots, q\,,$$

From equation (2.1.2), we have

$$\sum_{i=1}^{n_j}\sum_{k=1}^{p} y_{ijk} - n_j p\hat{\mu} - n_j p\hat{\alpha}_j = 0 \;\;, \forall j = 1, \dots, q\;\;,$$

$$\hat{\alpha}_j \; = \frac{1}{n_j p}\left(\sum_{i=1}^{n_j}\sum_{k=1}^{p} y_{ijk} - n_j p\hat{\mu}\right), \forall j = 1, \dots, q\,,$$

$$\hat{\alpha}_j = \tfrac{1}{n_j p}\textstyle\sum_{i=1}^{n_j}\sum_{k=1}^{p} y_{ijk} - \hat{\mu}, \forall j = 1, \dots, q,$$

$$\therefore\; \hat{\alpha}_j = \bar{y}_{.j.} - \bar{y}_{...} \;\;, \tag{2.2.3}$$

Now differentiate (2.2.1)with respect to β_k and equate the derivative to zero given us

$$\frac{\partial Q}{\partial \beta_k} = -2p\sum_{j=1}^{q}\sum_{i=1}^{n_j}\left(y_{ijk} - \mu - \alpha_j - \beta_k - (\alpha\beta)_{jk}\right) = 0, \forall k = 1, \dots, p,$$

$$\sum_{j=1}^{q}\sum_{i=1}^{n_j} y_{ijk} - n\hat{\mu} - \sum_{j=1}^{q} n_j\alpha_j - qn\hat{\beta}_k - \sum_{j=1}^{q} n_j(\alpha\beta)_{jk} = 0, \forall k = 1, \dots, p,$$

From equation (2.1.2), we have

$$\sum_{j=1}^{q}\sum_{i=1}^{n_j} y_{ijk} - n\hat{\mu} - n\hat{\beta}_k = 0, \forall k = 1, \dots, p,$$

$$\hat{\beta}_k = \frac{1}{n}\left(\sum_{j=1}^{q}\sum_{i=1}^{n_j} y_{ijk} - n\hat{\mu}\right), \forall k = 1, \dots, p,$$

$$\hat{\beta}_k = \frac{1}{n}\sum_{j=1}^{q}\sum_{i=1}^{n_j} y_{ijk} - \hat{\mu} \;\;, \forall k = 1, \dots, p,$$

$$\therefore\; \hat{\beta}_k = \bar{y}_{..k} - \bar{y}_{...}\,, \tag{2.2.4}$$

Now differentiate (2.2.1)with respect to $(\alpha\beta)_{jk}$ and equate the derivative to zero given us

$\frac{\partial Q}{\partial(\alpha\beta)_{jk}} = -2qp\sum_{i=1}^{n_j}(y_{ijk} - \mu - \alpha_j - \beta_k - (\alpha\beta)_{jk}) = 0$,

$$\sum_{i=1}^{n_j} y_{ijk} - n_j\hat{\mu} - n_j\hat{\alpha}_j - n_j\hat{\beta}_k - n_j\left(\widehat{\alpha\beta}\right)_{jk} = 0 , \forall j = 1, \dots, q \ \& \ \forall k = 1, \dots, p,$$

$$\left(\widehat{\alpha\beta}\right)_{jk} = \frac{1}{n_j}\left(\sum_{i=1}^{n_j} y_{ijk} - n_j\hat{\mu} - n_j\hat{\alpha}_j - n_j\hat{\beta}_k\right), \forall j = 1, \dots, q \ \& \ \forall k = 1, \dots, p,$$

$$(\widehat{\alpha\beta})_{jk} = \frac{1}{n_j}\sum_{i=1}^{n_j} y_{ijk} - \hat{\mu} - \hat{\alpha}_j - \hat{\beta}_k, \forall j = 1, \dots, q \ \& \ \forall k = 1, \dots, p,$$

$$\therefore \ (\widehat{\alpha\beta})_{jk} = \bar{y}_{.jk} - \bar{y}_{.j.} - \bar{y}_{..k} + \bar{y}_{...} \ , \qquad (2.2.5)$$

See table (3).

2.3. Expectation of Mean Squares

The model (2.1.1) can be write as

$$y_{ijk} - \mu = \alpha_j + \beta_k + (\alpha\beta)_{jk} + \delta_{i(j)} + e_{ijk} \ . \qquad (2.3.1)$$

We can replaced all the effects in this model by their estimates, then:

$$y_{ijk} - \bar{y}_{...} = \left(\bar{y}_{.j.} - \bar{y}_{...}\right) + (\bar{y}_{..k} - \bar{y}_{...}) + \left(\bar{y}_{ij.} - \bar{y}_{.j.}\right) + \left(\bar{y}_{.jk} - \bar{y}_{.j.} - \bar{y}_{..k} + \bar{y}_{...}\right) + \left(y_{ijk} - \bar{y}_{.jk} - \bar{y}_{ij.} + \bar{y}_{.j.}\right),$$

where

$$\left.\begin{aligned}
\bar{y}_{...} &= \frac{1}{np}\sum_{j=1}^{q}\sum_{i=1}^{n_j}\sum_{k=1}^{p} y_{ijk} \quad \textit{is the overall mean},\\
\bar{y}_{.j.} &= \frac{1}{n_j p}\sum_{i=1}^{n_j}\sum_{k=1}^{p} y_{ijk} \ , \forall\, j = 1, \dots, q\,, \ \textit{is the mean for group } j\ ,\\
\bar{y}_{..k} &= \frac{1}{n}\sum_{j=1}^{q}\sum_{i=1}^{n_j} y_{ijk} \ \textit{is the mean for time } k\ ,\\
\bar{y}_{.jk} &= \frac{1}{n_j}\sum_{i=1}^{n_j} y_{ijk} \ , \forall j = 1, \dots, q\ , \textit{is the mean for the group } j \textit{ at time } k\ ,\\
\bar{y}_{ij.} &= \frac{1}{p}\sum_{k=1}^{p} y_{ijk} \ , \textit{is a mean for the } i^{th} \textit{ subject in group } j\ ,
\end{aligned}\right\} \qquad (2.3.2)$$

It's important to learn about the properties of sum of squares in order to analyze how the model changes depending on the observed data:

The sums of squares Factor Between the units (Group) we denoted by

$$SS\alpha = SS(\alpha/\mu\,,\alpha,\alpha\beta,\delta) = SS(\mu,\alpha,\beta,(\alpha\beta),\delta) - SS(\mu,\beta,(\alpha\beta),\delta)\,,$$

$$SS\alpha = \sum_{j=1}^{q}\frac{y_{j.}^{2}}{n_j p} - \frac{y_{..}^{2}}{np} = p\sum_{j=1}^{q} n_j(\bar{y}_{.j.} - \bar{y}_{...})^2,$$

The sums of squares random effect SSδ

$$\mathrm{SS\delta} = \mathrm{SS}\big(\delta/\mu\,,\alpha,\beta,(\alpha\beta)\big) = \mathrm{SS}(\mu,\alpha,\beta,(\alpha\beta),\delta) - \mathrm{SS}\big(\mu,\alpha,\beta,(\alpha\beta)\big)\,,$$

$$SS\delta = P\sum_{j=1}^{q}\sum_{i=1}^{n_j}(\bar{y}_{ij.} - \bar{y}_{.j.})^2 \;\;,$$

The sums of squares of the Interaction between the between units (Group) and the Factor within the units (Time) we denoted by SSαβ

$$SS\alpha\beta = SS(\alpha\beta/\mu\,,\alpha,\beta,\delta) = SS(\mu,\alpha,\beta,(\alpha\beta),\delta) - SS(\mu,\alpha,\beta,\delta),$$

$$SS\alpha\beta = \sum_{j=1}^{q}\sum_{k=1}^{p}\frac{y_{.jk}^{2}}{n_j} - \sum_{j=1}^{q}\frac{y_{.j.}^{2}}{n_j p} - \sum_{k=1}^{p}\frac{y_{..k}^{2}}{n} + \frac{y_{...}^{2}}{np},$$

$$= \textstyle\sum_{j=1}^{q}\sum_{k=1}^{p} n_j(\bar{y}_{.jk} - \bar{y}_{.j.} - \bar{y}_{..k} + \bar{y}_{...})^2,$$

The sums of squares Factor within the units (Time) we denoted by SSβ

$$SS\beta = SS(\beta/\mu\,,\alpha,\alpha\beta,\delta) = SS(\mu,\alpha,\beta,(\alpha\beta),\delta) - SS(\mu,\alpha,(\alpha\beta),\delta)\,,$$

$$SS\beta = \textstyle\sum_{K=1}^{P}\frac{y_{..k}^{2}}{n} - \frac{y_{...}^{2}}{np} = n\sum_{K=1}^{P}(\bar{y}_{..k} - \bar{y}_{...})^2\,,$$

The error sum of squares SSe

$$SSe = \sum_{j=1}^{q}\sum_{i=1}^{n_j}\sum_{k=1}^{p}(y_{ijk} - \bar{y}_{.jk} - \bar{y}_{ij.} + \bar{y}_{j.})^2\,,$$

Now the sum of square to group, subjects (group),time, group × time (interaction) and residuals are given by:

$$\left.\begin{aligned}
SS\alpha &= P\sum_{j=1}^{q} n_j\left(\bar{y}_{.j.}-\bar{y}_{...}\right)^2, \\
SS\beta &= n\sum_{k=1}^{p}(\bar{y}_{..k}-\bar{y}_{...})^2 \ , \\
SS\delta &= p\sum_{j=1}^{q}\sum_{i=1}^{n_j}\left(\bar{y}_{ij.}-\bar{y}_{.j.}\right)^2, \\
SS\alpha\beta &= \sum_{j=1}^{q}\sum_{k=1}^{p} n_j(\bar{y}_{.jk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2, \\
SSe &= \sum_{j=1}^{q}\sum_{i=1}^{n_j}\sum_{k=1}^{p}\left(y_{ijk}-\bar{y}_{.jk}-\bar{y}_{ij.}+\bar{y}_{.j.}\right)^2.
\end{aligned}\right\} \qquad (2.3.3)$$

Then the total of variation can be written as

$$\mathrm{SST} = \mathrm{SS\alpha} + \mathrm{SS\beta} + \mathrm{SS\delta} + \mathrm{SS\alpha\beta} + \mathrm{SSe} \quad , \qquad (2.3.4)$$

and the corresponding degree of freedom is decomposed into

$$\mathrm{dfT} = \mathrm{df\alpha} + \mathrm{df\beta} + \mathrm{df\delta} + \mathrm{df\alpha\beta} + \mathrm{dfe} \ .$$

$$\therefore dfT = (q-1)+(n-q)+(p-1)+(q-1)(p-1)+(n-q)(p-1)$$

$$= np-1 \ . \qquad (\ 2.3.5\ )$$

Theorem (2.3.1)

The expectation of (SSβ),(SSα),(SSαβ),(SSδ), (SSe) are

$$n\sum_{k=1}^{p}(\beta_k^2)+(p-1)\sigma_e^2 \quad , \qquad pn\sum_{j=1}^{q}(\alpha_j^2)+p(q-1)\sigma_\delta^2+(q-1)\sigma_e^2$$

$$, n\sum_{j=1}^{q}\sum_{k=1}^{p}(\alpha\beta)_{jk}^2+(p-1)(q-1)\sigma_e^2 , \qquad p(n-q)\sigma_\delta^2 + \ (n-q)\sigma_e^2,$$

and $(p-1)(n-q)\sigma_e^2$ respectively.

Proof:

$$E(SS\beta) = E\left(n\sum_{k=1}^{p}(\bar{y}_{..k} - \bar{y}_{...})^2\right),$$

$$= E\left(\left(n\sum_{k=1}^{p}\begin{pmatrix}\mu + \bar{\alpha}_{.} + \beta_k + \left(\overline{\alpha\beta}\right)_{.k} + \bar{\delta}_{.(.)} + \bar{e}_{..k} - \mu - \\ \bar{\alpha}_{.} - \bar{\beta}_{.} - \left(\overline{\alpha\beta}\right)_{..} - \bar{\delta}_{.(.)} - \bar{e}_{...}\end{pmatrix}^2\right)\right),$$

$$\because\ \bar{\alpha}_{.} = 0, \qquad \bar{\beta}_{.} = 0\ ,\ \left(\overline{\alpha\beta}\right)_{..} = 0\ ,\ \left(\overline{\alpha\beta}\right)_{.k} = 0 \quad ,$$

$$= E\left(n\sum_{k=1}^{p}(\beta_k + \bar{e}_{..k} - \bar{e}_{...})^2\right),$$

$$= E\left(n\sum_{k=1}^{p}\left(\beta_k^2 + (\bar{e}_{..k} - \bar{e}_{...})^2\right)\right),$$

$$= n\sum_{k=1}^{p}(\beta_k^2) + n\sum_{k=1}^{p}\frac{(p-1)\sigma_e^2}{np}\ ,$$

$$= n\sum_{k=1}^{p}(\beta_k^2) + (p-1)\sigma_e^2\ .$$

$$E(SS\alpha) = E\left(P\sum_{j=1}^{q}n_j\left(\bar{y}_{.j.} - \bar{y}_{...}\right)^2\right),$$

$$= E\left(P\sum_{j=1}^{q}n_j\begin{pmatrix}\mu + \alpha_j + \bar{\beta}_{.} + \left(\overline{\alpha\beta}\right)_{.j} + \bar{\delta}_{.(j)} + \bar{e}_{.j.} - \mu - \\ \bar{\alpha}_{.} - \bar{\beta}_{.} - \left(\overline{\alpha\beta}\right)_{..} - \bar{\delta}_{.(.)} - \bar{e}_{...}\end{pmatrix}^2\right),$$

$$\because\ \bar{\alpha}_{.} = 0, \qquad \bar{\beta}_{.} = 0\ ,\ \left(\overline{\alpha\beta}\right)_{...} = 0\ ,\ \left(\overline{\alpha\beta}\right)_{.j.} = 0\ ,$$

$$= E\left(P\sum_{j=1}^{q}n_j\left(\alpha_j + \bar{\delta}_{.(j)} + \bar{e}_{.j.} - \bar{\delta}_{.(.)} - \bar{e}_{...}\right)^2\right),$$

$$= E\left(P\sum_{j=1}^{q}n_j\left(\alpha_j^2 + \left(\bar{\delta}_{.(j)} - \bar{\delta}_{.(.)}\right)^2 + \left(\bar{e}_{.j.} - \bar{e}_{...}\right)^2\right)\right),$$

$$= p\sum_{j=1}^{q}n_j(\alpha_j^2) + np\frac{(q-1)\sigma_\delta^2}{n} + np\frac{(q-1)\sigma_e^2}{np},$$

$$= pn\sum_{j=1}^{q}(\alpha_j^2) + p(q-1)\sigma_\delta^2 + (q-1)\sigma_e^2\ .$$

$$E(SS\alpha\beta) = E\sum_{j=1}^{q}\sum_{k=1}^{p}n_j(\bar{y}_{.jk} - \bar{y}_{.j.} - \bar{y}_{..k} + \bar{y}_{...})^2\ ,$$

$$= E\sum_{j=1}^{q}\sum_{k=1}^{p}\begin{matrix}n_j(\mu + \alpha_j + \beta_k + (\alpha\beta)_{jk} + \bar{\delta}_{.(j)} + \bar{e}_{.jk} - \\ (\mu + \alpha_j + \bar{\beta}_{.} + \left(\overline{\alpha\beta}\right)_{j.} + \bar{\delta}_{.(j)} + \bar{e}_{.j.})\end{matrix}$$

$$-\left(\mu + \bar{\alpha}_{.} + \beta_k + \left(\overline{\alpha\beta}\right)_{.k} + \bar{\delta}_{.(.)} + \bar{e}_{..k}\right) + (\mu + \bar{\alpha}_{.} + \bar{\beta}_{.} + \left(\overline{\alpha\beta}\right)_{..} + \bar{\delta}_{.(.)} + \bar{e}_{...}))^2,$$

$$= E\sum_{j=1}^{q}\sum_{k=1}^{p}n_j((\alpha\beta)_{jk} + \left(\bar{e}_{.jk} - \bar{e}_{.j.}\right) + (\bar{e}_{...} - \bar{e}_{..k}))^2,$$

$$= n\sum_{j=1}^{q}\sum_{k=1}^{p}(\alpha\beta)_{jk}^2 + (p-1)(q-1)\sigma_e^2\ .$$

$$E(SS\delta) = E(p\sum_{j=1}^{q}\sum_{i=1}^{n_j}\left(\bar{y}_{ij.} - \bar{y}_{.j.}\right)^2),$$

$$= p\sum_{j=1}^{q}\sum_{i=1}^{n_j} E\left(\begin{array}{c}\mu + \alpha_j + \bar{\beta}_{.} + \left(\overline{\alpha\beta}\right)_{j.} + \delta_{i(j)} + \bar{e}_{ij.} \\ -\left(\mu + \alpha_j + \bar{\beta}_{.} + \left(\overline{\alpha\beta}\right)_{j.} + \bar{\delta}_{.(j)} + \bar{e}_{.j.}\right)\end{array}\right)^2,$$

$$= p\sum_{j=1}^{q}\sum_{i=1}^{n_j} E(\left(\delta_{i(j)} + \bar{e}_{ij.} - \left(\bar{\delta}_{.(j)} + \bar{e}_{.j.}\right)\right)^2),$$

$$= p\sum_{j=1}^{q}\sum_{i=1}^{n_j} E(\left((\delta_{i(j)} - \bar{\delta}_{.(j)}) + (\bar{e}_{ij.} - \bar{e}_{.j.})\right)^2),$$

$$= p(n-q)\sigma_\delta^2 + (n-q)\sigma_e^2 .$$

$$E(SSe) = \sum_{j=1}^{q}\sum_{i=1}^{n_j}\sum_{k=1}^{p} E\left(y_{ijk} - \bar{y}_{.jk} - \bar{y}_{ij.} + \bar{y}_{.j.}\right)^2,$$

$$= \sum_{j=1}^{q}\sum_{i=1}^{n_j}\sum_{k=1}^{p} E\left(e_{ijk} - \bar{e}_{.jk} - \bar{e}_{ij.} + \bar{e}_{.j.}\right)^2,$$

$$= \sum_{j=1}^{q}\sum_{i=1}^{n_j}\sum_{k=1}^{p} E\left(\left(e_{ijk} - \bar{e}_{.jk}\right) + \left(\bar{e}_{.j.} - \bar{e}_{ij.}\right)\right)^2,$$

$$= (p-1)(n-q)\sigma_e^2 .$$

□

Theorem (2.3.2)

The expectations of (MSα) , (MSβ), (MSαβ), (MSδ), and (MSe) are

$$\frac{pn}{q-1}\sum_{j=1}^{q}(\alpha_j^2) + p\sigma_\delta^2 + \sigma_e^2 \quad , \quad \frac{n}{p-1}\sum_{k=1}^{p}(\beta_k^2) + \sigma_e^2 \quad ,$$

$\frac{n}{(q-1)(p-1)}\sum_{j=1}^{q}\sum_{k=1}^{p}(\alpha\beta)_{jk}^2 + \sigma_e^2$, $P\sigma_\delta^2 + \sigma_e^2$, σ_e^2 respectively.

Proof:

$$E(MS\alpha) = E\left(\frac{SS\alpha}{(q-1)}\right) = \frac{1}{(q-1)}E(SS\alpha) \quad ,$$

$$= \frac{1}{(q-1)}(pn\sum_{j=1}^{q}(\alpha_j^2) + p(q-1)\sigma_\delta^2 + (q-1)\sigma_e^2) \quad ,$$

$$= \frac{pn}{q-1}\sum_{j=1}^{q}(\alpha_j^2) + p\sigma_\delta^2 + \sigma_e^2 .$$

$$E(MS\beta) = E\left(\frac{SS\beta}{p-1}\right) = \frac{1}{P-1}E(SS\beta) = \frac{1}{P-1}\left(n\sum_{k=1}^{p}(\beta_k^2) + (p-1)\sigma_e^2\right) \quad ,$$

$$= \frac{n}{p-1}\sum_{k=1}^{p}(\beta_k^2) + \sigma_e^2 .$$

$$E(MS\alpha\beta) = E\left(\frac{SS\alpha\beta}{(q-1)(p-1)}\right) = \frac{1}{(q-1)(p-1)} E(SS\alpha\beta)\,,$$
$$= \frac{1}{(q-1)(p-1)}\left(n\sum_{j=1}^{q}\sum_{k=1}^{p}(\alpha\beta)_{jk}^2 + (p-1)(q-1)\sigma_e^2\right),$$
$$= \frac{n}{(q-1)(p-1)}\sum_{j=1}^{q}\sum_{k=1}^{p}(\alpha\beta)_{jk}^2 + \sigma_e^2\,.$$

$$E(MS\delta) = E\left(\frac{SS\delta}{n-q}\right) = \frac{1}{n-q} E(SS\delta)\,,$$
$$= \frac{1}{n-q}\left(p(n-q)\sigma_\delta^2 + (n-q)\sigma_e^2\right) = P\sigma_\delta^2 + \sigma_e^2\,.$$

$$E(MSe) = E\left(\frac{SSe}{(p-1)(n-q)}\right) = \frac{1}{(p-1)(n-q)} E(SSe)\,,$$
$$= \frac{1}{(p-1)(n-q)}(p-1)(n-q)\sigma_e^2 = \sigma_e^2\,.$$

□

2.4 Analysis of Variance Table

In this section, we state the analysis of variance (ANOVA) table for the model including the results over, with the degrees of freedom(d.f),mean square(MS), and expectation of mean square (E(MS)) as follows:

Table (1) ANOVA Table for the repeated measurements model with random effects for unbalanced data

Source of variance	d.f	SS	MS	E(MS)
Group	$q-1$	$SS\alpha$	$\frac{SS\alpha}{q-1}$	$\frac{p}{q-1}\sum_{j=1}^{q} n_j\,\alpha_j^2 + p\sigma_\delta^2 + \sigma_e^2$
Unit(Group)	$n-q$	$SS\delta$	$\frac{SS\delta}{n-q}$	$p\sigma_\delta^2 + \sigma_e^2$
Time	p-1	$SS\beta$	$\frac{SS\beta}{p-1}$	$\frac{q}{p-1}\sum_{k=1}^{p} n_j\beta_k^2 + \sigma_e^2$
Group ×Time	$(q-1)(p-1)$	$SS\alpha\beta$	$\frac{SS\alpha\beta}{(q-1)(p-1)}$	$\frac{1}{(p-1)(q-1)}\sum_{j=1}^{q}\sum_{k}^{p} n_j(\alpha\beta)_{jk}^2 + \sigma_e^2$
Residual	$(p-1)(n-q)$	SSe	$\frac{SSe}{(p-1)(n-q)}$	σ_e^2
Total	$np-1$	SST		

2.4.1. Properties of the Estimators

We introduce some properties of Ordinary Least Squares estimators by theorems

Theorem (2.4.1)

The estimators $\hat{\mu}$, $\hat{\alpha}_j$, $\hat{\beta}_k$, *and* $\widehat{(\alpha\beta)}_{jk}$ are unbiased of μ , α_j , β_k , *and* $(\alpha\beta)_{jk}$ respectively.

Proof:

From equation (2.2.2) and (2.3.2) we have

$E(\hat{\mu}) = E(\bar{y}_{...})$,

$= \frac{1}{np} E\left(\sum_{j=1}^{q} \sum_{i=1}^{n_j} \sum_{k=1}^{p} y_{ijk}\right)$,

$= \frac{1}{np} E\left(\sum_{j=1}^{q} \sum_{i=1}^{n_j} \sum_{k=1}^{p} \left(\mu + \alpha_j + \beta_k + (\alpha\beta)_{jk} + \delta_{i(j)} + e_{ijk}\right)\right)$,

and from equation (2.1.2)

$E(\hat{\mu}) = \frac{1}{np}(np\mu) = \mu$,

$\therefore$ $\hat{\mu}$ is unbiased estimator of μ.

Now , from equation (2.2.3) and (2.3.2) we have

$E(\hat{\alpha}_j) = E(\bar{y}_{.j.} - \bar{y}_{...})$,

$= E(\bar{y}_{.j.}) - E(\bar{y}_{...})$,

$= \frac{1}{n_j p} E\left(\sum_{i=1}^{n_j} \sum_{k=1}^{p} y_{ijk}\right) - \mu$,

$= \frac{1}{n_j p} E\left(\sum_{i=1}^{n_j} \sum_{k=1}^{p} \left(\mu + \alpha_j + \beta_k + (\alpha\beta)_{jk} + \delta_{i(j)} + e_{ijk}\right)\right) - \mu$,

and from equation (2.1.2)

$= \frac{1}{n_j p} E\left(n_j p\mu + n_j p\alpha_j\right) - \mu$,

$= \mu + \alpha_j - \mu = \alpha_j$,

$\therefore$ $\hat{\alpha}_j$ is unbiased estimator of α_j .

Now , from equation (2.2.4) and (2.3.2) we have

$E(\hat{\beta}_k) = E(\bar{y}_{..k} - \bar{y}_{...})$,

$= E(\bar{y}_{..k}) - E(\bar{y}_{...})$,

$= \frac{1}{n} E\left(\sum_{j=1}^{q} \sum_{i=1}^{n_j} y_{ijk}\right) - \mu$,

$= \frac{1}{n} E\left(\sum_{j=1}^{q} \sum_{i=1}^{n_j} \left(\mu + \alpha_j + \beta_k + (\alpha\beta)_{jk} + \delta_{i(j)} + e_{ijk}\right)\right) - \mu$,

and from equation (2.1.2)

$= \frac{1}{n} E(n\mu + n\beta_k) - \mu$,

$=\mu+\beta_k-\mu=\beta_k$,

$\therefore$ $\hat{\beta}_k$ is unbiased estimator of β_k .

Now , from equation (2.2.5) and (2.3.2) we have

$$E\left(\left(\widehat{\alpha\beta}\right)_{jk}\right)=E\left(\bar{y}_{.jk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...}\right),$$

$$E\left(\left(\widehat{\alpha\beta}\right)_{jk}\right)=E\left(\bar{y}_{.jk}\right)-E(\bar{y}_{.j.})-E(\bar{y}_{..k})+E(\bar{y}_{...}) ,$$

$$=\frac{1}{n_j}E\left(\sum_{i=1}^{n_j}y_{ijk}\right)-\frac{1}{n_jp}E\left(\sum_{i=1}^{n_j}\sum_{k=1}^{p}y_{ijk}\right)-\frac{1}{n}E\left(\sum_{j=1}^{q}\sum_{i=1}^{n_j}y_{ijk}\right)+\mu ,$$

$$=\frac{1}{n_j}E\left(\sum_{i=1}^{n_j}(\mu+\alpha_j+\beta_k+(\alpha\beta)_{jk}+\delta_{i(j)}+e_{ijk})\right)-\frac{1}{n_jp}E\left(\sum_{i=1}^{n_j}\sum_{k=1}^{p}\left(\mu+\alpha_j+\beta_k+(\alpha\beta)_{jk}+\delta_{i(j)}+e_{ijk}\right)\right)-\frac{1}{n}E\left(\sum_{j=1}^{q}\sum_{i=1}^{n_j}\left(\mu+\alpha_j+\beta_k+(\alpha\beta)_{jk}+\delta_{i(j)}+e_{ijk}\right)\right)+\mu ,$$

and from equation (2.1.2)

$$=\frac{1}{n_j}E\left(n_j\mu+n_j\alpha_j+n_j\beta_k+n_j(\alpha\beta)_{jk}\right)-\frac{1}{n_jp}E\left(n_jp\mu+n_jp\alpha_j\right)-\frac{1}{n}E(n\mu+n\beta_k)+\mu ,$$

$$=\left(\mu+\alpha_j+\beta_k+(\alpha\beta)_{jk}\right)-\left(\mu+\alpha_j\right)-(\mu+\beta_k)+\mu=(\alpha\beta)_{jk} ,$$

$\therefore$ $\left(\widehat{\alpha\beta}\right)_{jk}$ is unbiased estimator of $(\alpha\beta)_{jk}$. □

Theorem (2.4.2)

The estimators $\hat{\mu}$, $\hat{\alpha}_j$, $\hat{\beta}_k$, *and* $\left(\widehat{\alpha\beta}\right)_{jk}$ are consistent of μ , α_j , β_k , *and* $(\alpha\beta)_{jk}$ respectively.

Proof:

$\because$ $\hat{\mu}=\bar{y}_{...}\Longrightarrow var(\hat{\mu})=var(\bar{y}_{...})$,

$$var(\hat{\mu})=var\left(\frac{1}{np}\sum_{j=1}^{q}\sum_{i=1}^{n_j}\sum_{k=1}^{p}y_{ijk}\right) ,$$

$$=\frac{1}{n^2p^2}var\left(\sum_{j=1}^{q}\sum_{i=1}^{n_j}\sum_{k=1}^{p}y_{ijk}\right),$$

$$=\frac{1}{n^2p^2}(np)\left(\sigma_\delta^2+\sigma_e^2\right) ,$$

$$=\frac{\sigma_\delta^2+\sigma_e^2}{np},$$

$$=\frac{\sigma_\delta^2+\sigma_e^2}{m}\longrightarrow 0\ as\ \ m\longrightarrow\infty\ ,where\ m=np .$$

$\therefore$ $\hat{\mu}$ is consistent of μ.

$\because\ \hat{\alpha}_j = \bar{y}_{.j.} - \bar{y}_{...}$,

$var(\hat{\alpha}_j) = var(\bar{y}_{.j.} - \bar{y}_{...})$,

$var(\hat{\alpha}_j) = var(\bar{y}_{.j.}) - var(\bar{y}_{...})$,

$$= var\left(\frac{1}{n_j p}\sum_{i=1}^{n_j}\sum_{k=1}^{p} y_{ijk}\right) - \frac{\sigma_\delta^2+\sigma_e^2}{np},$$

$$= \frac{1}{n_j^2 p^2}(n_j p)(\sigma_\delta^2 + \sigma_e^2) - \frac{\sigma_\delta^2+\sigma_e^2}{np} ,$$

$$= \frac{\sigma_\delta^2+\sigma_e^2}{n_j p} - \frac{\sigma_\delta^2+\sigma_e^2}{np} = \frac{(n-n_j)(\sigma_\delta^2+\sigma_e^2)}{n_j np} ,$$

$$= \frac{(n-n_j)(\sigma_\delta^2+\sigma_e^2)}{m_1} \to 0 \ \ as\ \ m_1 \to \infty$$, where $m_1 = n_j np$.

$\therefore\ \hat{\alpha}_j$ is consistent of α_j .

$\because\ \hat{\beta}_k = \bar{y}_{..k} - \bar{y}_{...}$,

$var(\hat{\beta}_k) = var(\bar{y}_{..k} - \bar{y}_{...})$,

$$= var(\bar{y}_{..k}) - var(\bar{y}_{...}) ,$$

$$= var\left(\frac{1}{n}\sum_{j=1}^{q}\sum_{i=1}^{n_j} y_{ijk}\right) - \frac{\sigma_\delta^2+\sigma_e^2}{np} ,$$

$$= \frac{1}{n^2} var\left(\sum_{j=1}^{q}\sum_{i=1}^{n_j} y_{ijk}\right) - \frac{\sigma_\delta^2+\sigma_e^2}{np} ,$$

$$= \frac{1}{n^2}(n)(\sigma_\delta^2 + \sigma_e^2) - \frac{\sigma_\delta^2+\sigma_e^2}{np},$$

$$= \frac{\sigma_\delta^2+\sigma_e^2}{n} - \frac{\sigma_\delta^2+\sigma_e^2}{np} = \frac{(p-1)(\sigma_\delta^2+\sigma_e^2)}{np} ,$$

$$= \frac{(p-1)(\sigma_\delta^2+\sigma_e^2)}{m} \to 0 \ \ as\ \ m \to \infty ,$$

$\therefore\ \hat{\beta}_k$ is consistent of β_k .

$\because\ (\widehat{\alpha\beta})_{jk} = \bar{y}_{.jk} - \bar{y}_{.j.} - \bar{y}_{..k} + \bar{y}_{...}$,

$var((\widehat{\alpha\beta})_{jk}) = var(\bar{y}_{.jk} - \bar{y}_{.j.} - \bar{y}_{..k} + \bar{y}_{...})$,

$$= var(\bar{y}_{.jk}) - var(\bar{y}_{.j.}) - var(\bar{y}_{..k}) + var(\bar{y}_{...}),$$

$$= var\left(\frac{1}{n_j}\sum_{i=1}^{n_j} y_{ijk}\right) - \frac{\sigma_\delta^2+\sigma_e^2}{n_j p} - \frac{\sigma_\delta^2+\sigma_e^2}{n} - \frac{\sigma_\delta^2+\sigma_e^2}{np} ,$$

$$= \frac{1}{n_j^2}(n_j)(\sigma_\delta^2 + \sigma_e^2) - \frac{\sigma_\delta^2+\sigma_e^2}{n_j p} - \frac{\sigma_\delta^2+\sigma_e^2}{n} - \frac{\sigma_\delta^2+\sigma_e^2}{np} \ ,$$

$$= \frac{(\sigma_\delta^2+\sigma_e^2)}{n_j} - \frac{\sigma_\delta^2+\sigma_e^2}{n_j p} - \frac{\sigma_\delta^2+\sigma_e^2}{n} - \frac{\sigma_\delta^2+\sigma_e^2}{np} \ ,$$

$$= \frac{(p-1)(n-n_j)\sigma_\delta^2+\sigma_e^2}{n n_j p} \ ,$$

$$= \frac{(p-1)(n-n_j)\sigma_\delta^2+\sigma_e^2}{m_1} \rightarrow \infty \ as \ m_1 \rightarrow 0 \ ,$$

$\therefore \ (\widehat{\alpha\beta})_{jk}$ is consistent of $(\alpha\beta)_{jk}$.

□

3. The Practical Side of the Study

In this section, we check the formulas and theories reached on the theoretical side have been taken a realistic experiment from the Palm Research Center at Basra University, during the two growing seasons 2019-2020 [7]. The objective of the experiment is to study the effect of the experimental factors and their interactions on the production of date palm, cultivar El Halawi. The experiment included three levels of unit factor, 3 days,6 days, and 9 days of the irrigation interval, also there are five levels of between unit factor, fruit length, fruit weight, fruit size, dry weight ,and total sugars ,and three levels of within unit factor, drip irrigation ,basin irrigation, and the regular tidal method.

According to the mathematical formula of the model study (2.1.1) and by applying the model to the experiment, we get the sum squares , of the effects between units factors, within-units factors, effect to interaction between-unit factor (Group) and within-units factors (Time), of the experiment as follows:

Table (2) ANOVA table for the data of experiment

Source of variance	d.f	SS	MS	F-Test
Group	4	61916.134	15479.0335	$F = \frac{MS\alpha}{MS\delta} = 2088.77^*$ $F_t(4,10,0.05) = 3.48$
Unit(Group)	10	74.106	7.4106	
Time	2	2.972	1.486	$F = \frac{MS\beta}{MSE} = 0.57^*$ $F_t(2,20,0.05) = 3.49$
Group× Time	8	2.103	0.262875	$F = \frac{MS\alpha\beta}{MSE} = 0.1007^*$ $F_t(8,20,0.05) = 2.45$
Residual	20	52.188	2.6094	
Total	44	62047.503		

Table (2) shows the results for the analysis of variance for model, we notice that the irrigation intervals has an important influence for the productive qualities were the calculated F-values is (2088.77) more than the tabulated F-values at 0.05 level significant .

Table (3) estimation values for the parameters of the model(2.1.1) for the experiment

$\hat{\mu}$	$\hat{\alpha}_j$	$\hat{\beta}_k$	$\widehat{(\alpha\beta)}_{jk}$	$\hat{\sigma}_\delta^2$	$\hat{\sigma}_e^2$
34.155	136.619	68.309	273.237	2.374	1.450

Table (3) above show the estimation values for the parameters of the model from the least squares method where we notice through these values of the estimators that they are encouraging values from a statistical point of view.

4. Conclusions

In this paper, we obtained some conclusions as follows:

1- The least squares estimators of parameters of repeated measurements model are $\hat{\mu} = \bar{y}_{...}$, $\hat{\alpha}_j = \bar{y}_{.j.} - \bar{y}_{...}$, $\hat{\beta}_k = \bar{y}_{..k} - \bar{y}_{...}$,and $\widehat{(\alpha\beta)}_{jk} = \bar{y}_{.jk} - \bar{y}_{.j.} - \bar{y}_{..k} + \bar{y}_{...}$.
2- The least squares estimators of parameters of repeated measurements model are unbiased and consistent.
3- The studied irrigation methods and irrigation interval an effect for improving productive qualities.

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