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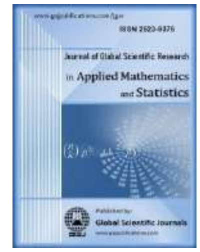

CoSemiprime Ring

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we introduce a new type of rings,namely cosemiprime ringas a generalization of semiprime ring, where a ring C is said to be cosemiprime ring if I=$\sqrt{I}$ for each ideal I of C .Some properties of this type of rings are obtained . Also, we determin the relationship among types of ideals in this ring.

1. Introduction

Let R be a ring with unity .Anon zero ring R is called prime ring if and only if for any a and b of R, abr=0 for all r in R implies that either a=0 or b=0 [1].In[2] the concept of prime ideal was introduced by Dummit, David, Foot, where an ideal I of a ring R is called prime if for every a,b in R ;ab∈I implies either a∈I or b∈I.In 1992 Park, Younk Soo and Kim, Ju.Pil introduced the concept of semiprime ideal, where an ideal I of ring R is called semiprime ideal if for each y in R;if y2∈I implies y∈I[3], equivalently an ideal I is called semiprime ideal if Jn ⊆ I for some ideal J of R and some positive integer n ,then J ⊆ I [4].Lam,Tsit-Yuea introduced in [5]a ring R is called semiprime ring if the zero ideal is a semiprime ideal .Let I be an ideal of ring R,then the nilradical of I denoted by $\sqrt{I}$,where $\sqrt{I}$ ={x ∈ R ; xn ∈ I} [6].That a nilradical of I need not be equal I for example : Let I =[4]in Z8 ,then $\sqrt{[4]}$=[2],so I ≠ $\sqrt{I}$.In this paper we study rings satisfying the condition $\sqrt{I}$ =I for each ideal I of this ring,which we called cosemiprime ring.We prove many properties for this kind of rings.Every zero diviser of C/I is a nilpotent element, where I be an ideal of Cosemiprime ring C.

This papercosists of two section. In section one, we study the basic properties of cosemiprime ring, in section two we devoted to study the generalization of the consepts:

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prime, primary, semiprime and semiprimary ideals in cosemiprime ring, we prove some theorems and give some examples to show the relationships among them.

2. Basic properties of Cosemiprime rings

In this section, we introduce the notion of cosemiprime ring as a generalization of semiprime ring and we study some properties of this kind of rings.

Definition (2.1):A ring C is called cosemiprime ring if I=$\sqrt{I}$ for each non zero ideal I of C.
Remarks and Examples (2.2).

1. It is clear that every ideal in cosemiprime ring is semiprime ideal.
2. Zn is cosemiprime ring if and only if n is a prime number.
   Proof: it is obviose.
3. Zn is cosemiprime ring if and only if n can be written as a product of two prime numbers.
   Proof:let n=p1p2 ; p1,p2 be prime number;I1=(P1) and I2=(P2),where $\sqrt{(p1)}$ =(p1) and $\sqrt{(p2)}$ =(p2) ,thus Zn is cosemiprime ring .
   Conversely :is obviose
4. It is clear that every simple ring is cosemiprime ring.
5. Every cosemiprime ring is ring ,but the converse is not true in general for examples(Z12+12 ,.12)is ring but not cosemiprime ring ,since there exist an ideal I=[4]of the ring(Z12,+12,.12) ,where $\sqrt{[4]}$ =[2].
6. It is clear that every sub cosemiprime ring is cosemiprime ring.
7. Z is not cosemiprime ring.
8. Every filed is cosemiprime ring.
   Proof:it is obviouse.
   But the converes is not true for example:(Z6,+6,.6) is cosemiprime ring,by remark and example (2.2-(3)),but it is not filed by[7].
9. Every prime ring is cosemiprime ring.
10. Proof:every prime ring is finite filed[7] and by remarks and example(2.2(8)) we obtain the result.
11. Recall that a subring B of a ring R is called a direct summand of R if and only if there exist a subring A of R such that R=B ⊕ A [8].

Proposition (2.3).Adirect summand of cosemiprime ring is a cosemiprime ring.

Proof:let C be a cosemiprime ring,let W be a subring of C ;C=W⨁X for subring X of C,then W is cosemiprime ring by[remarkes and examples 2.2(6)].

Corollary(2.4).the direct sum of two cosemiprime ring is a cosemiprime ring.

Proof:let C=R ⊕ W;R,W be two cosemiprime rings,let I be an ideal of R ,Y be an ideal of W,so (I,Y) be an ideal of C=R ⊕ W .Thus $\sqrt{(I,Y)}$ ={(a,b) ∈ C ;(a,b)n ∈ (I,Y)}={(a,b)∈C;(an,bn)∈(I,Y)}=($\sqrt{I}$,$\sqrt{Y}$)=(I,Y) ,since R,W be two cosemiprime rings.

Theorem (2.5).suppose that C,C1 be two rings and let f:C → C1 epimorphsim such that ker f ⊆ I,where I is an ideal of C,if C is cosemiprime ring , then C1 is cosemiprime ring.

Proof:let x ∈ $\sqrt{f(I)}$,so there exist n ∈ Z+;xn ∈ f(I),thus xn=f(y); y∈ I.Since f is onto,implies for each x ∈ C1 there exist a ∈ C;x=f(a).f(an −y)=0 , which mean (an-y)∈ ker f ⊆ I,implies (an −y) ∈ I , so an −y= i for some i ∈ I.Thus an=i+y ∈ I,so a ∈ $\sqrt{I}$,but x=f(a),so x∈ f($\sqrt{I}$)=f(I),which mean $\sqrt{f(I)}$ ⊆ f(I),but f(I) ⊆ $\sqrt{f(I)}$ [8].Which mean C1 is cosemiprime ring.

Corollary (2.6).the homomorphic image of a cosemiprime ring is cosemiprime ring.

Let f:C → C1 ;be homomorphism fuction,I1 be cosemiprime ideal of C1,let x ∈ $\sqrt{f^{-1}(I)}$,then xn∈f1(I1),so f(xn) ∈ I1,but f is homomorphism,so fn (x) ∈ I1,implies f(x) ∈ $\sqrt{I_1}$=I1so f(x) ∈ I1,thus x ∈ f-1 (I1),which mean $\sqrt{f^{-1}(I)}$ ⊆f-1(I),but f-1(I) ⊆ $\sqrt{f^{-1}(I)}$.Thus f-1 (I)=$\sqrt{f^{-1}(I)}$.

Nathan in[8]-[9] we call an ideal I is principle if is generated by single element .Thus a ∈ R is called principle ideal and denoted by (a)={ra;r ∈ R}=Ra,so a ring R is called principle ideal ring (P.I.R)if and only if every ideal of R is principle ideal.

Proposition (2.7).A cosemiprime rig is P.I.R.

Proof:let I be an ideal of cosemiprime ring C,if I={0} ,then I=(0).If I ≠{0},we must prove I=(a) ;a ∈C,a is the small positive integer in I.It is clear that (a) ⊆ I.Let x ∈ I=$\sqrt{I}$,so xn ∈ I,by division algorithm xn=aq +r ; 0< r <a ,implies r=xn − aq ∈ I ,which is a contradiction since a is a smallest element ,so r=0 , implies xn=aq.thus xn ∈ (a) ,so x ∈ $\sqrt{(a)}$ =(a).Thus I=(a).

But the coverse is not true for example: (Z,+,.) is P.I.R [8],but is not cosemiprime ring byremarks and examples(2.2-(7)).

3. Prime ideal,primary ideal , semiprime ideal and semiprimary ideal of cosemiprime ring

In this section we generalize the concept of (prime,primary,semiprime and semiprimary) ideal to (cosprime,cosprimary,cossemiprime and cosemiprimary)ideal in cosemiprime ring ,then we will discuss several propositions and theorems. .

Definition(3.1). an ideal I of cosemiprime ring C is called cosprime ideal if for each a,b in C ;ab∈ $\sqrt{I}$ implies either a∈ $\sqrt{I}$ or b∈ $\sqrt{I}$.

Theorem (3.2).suppose I be cosprime ideal of cosemiprime ring C ,then every zero diviser of C\I is a nilpotent element.

Proof:let a\=a+I be the zero diviser in C\I,so a\ ≠ $0_{C\backslash I}$ then there exist b\ ≠ $0_{C\backslash I}$; a\ b\ =$0_{C\backslash I}$.

a\ b\ =ab+I=I ,which mean ab∈ I=$\sqrt{I}$,so (ab)n ∈ I,so $a^n b^n$ ∈ I,but I is a cosprime ideal so either a^n ∈ I or b^n ∈ I.Thus $(a+I)^n$=I or $(b+I)^n$=I,which mean a\ is a nilpotent of C\I.

Definition (3.3) .an ideal J is called cosprimary ideal of cosemiprime ideal C if and only if ∀ a,b ∈C ;ab∈ $\sqrt{I}$ implies either a∈ $\sqrt{I}$ or bm ∈ $\sqrt{I}$ for some positive integer m.

Definition (3.4).an ideal J is called cossemiprime of a cosemiprime ring C if and only if for each a∈C,if a2n ∈I implies an ∈I.

Definition (3.5).let I be an ideal of a cosemiprime ring ,then I is called a cossemiprimary ideal of a cosemiprime ring C if and only if ∀x,y ∈C ;x.y ∈ $\sqrt{I}$ implies either xm ∈ $\sqrt{I}$ or ym ∈ $\sqrt{I}$ for some m∈Z+..

Proposition (3.6).let C be cosemiprime ring and I be cosprime ideal,then I is a cosprimary ideal of C.

Proof:let x,y ∈C,since I is cosprime ideal,then if xy∈ $\sqrt{I}$ implies either x∈ $\sqrt{I}$ or y∈ $\sqrt{I}$ which mean I is a cosprimary ideal.

Theorem (3.7).let C be a cosemiprime ring ,I is an ideal of C .Then I is a cosprime ideal of C if and only if it is cosprimary ideal.

Proof:→ is directly from proposition 3.6

Conversely,let x,y ∈C such that xy ∈ $\sqrt{I}$,since I is cosprimary ideal implies either x∈ $\sqrt{I}$ or ym∈ $\sqrt{I}$ for some m∈Z+.Thus either x∈ $\sqrt{I}$ or ymn∈ I,but I is ideal in cosemiprime ring ,so either x∈ $\sqrt{I}$ or y ∈ $\sqrt{I}$ for some mn ∈ Z+,which mean I is a coprime ideal.

Theorem (3.8).suppose C be cosemiprime ring,I be an ideal of C ,then I is coprimary ideal if and only if I is cosemiprime ideal.

Proof:let I be coprimary ideal,for each x,y ∈ C ;xy∈ $\sqrt{I}$ implies either x ∈ $\sqrt{I}$ or ym ∈ $\sqrt{I}$ .If x∉ $\sqrt{I}$,y2n∈ $\sqrt{I}$ (by definition 3.4), so y2n∈I,implies y2 ∈ $\sqrt{I}$,which mean y.y ∈ $\sqrt{I}$ so by theorem (2.7) either y ∈ $\sqrt{I}$ or y ∈ $\sqrt{I}$ which mean yn ∈I ,so I is cosemiprime ideal.Conversely ,if I is cosemiprime ideal to prove I is coprimary ideal ,if xy ∈ $\sqrt{I}$,suppose x∉ $\sqrt{I}$,if y2n ∈ $\sqrt{I}$,then y ∈ $\sqrt{I}$ which mean yn ∈I=$\sqrt{I}$,for some n∈ Z+.Thus I is coprimary ideal.

Theorem(3.9) .suppose C be cosemiprime ring ,I be an ideal of C .Then I is cosemiprimary ideal if and only if I is coprimary ideal.

Poof:it is clear that every coprimary ideal is cosemiprimary ideal.Conversely,if ab∈ $\sqrt{I}$ such that anbn ∈ I,but I is cosemiprimary ideal so either an ∈ $\sqrt{I}$ or bn ∈ $\sqrt{I}$ which is mean is coprimary.

Corollary(3.10).let C be cosemiprime ring,I be an ideal of C .Then I is cosemiprimary ideal if and only if I is cosemiprime ideal.

Proof:is directly from theorem (3.8)and theorem (3.9).

Corollary (3.11).suppose C be cosemiprime ring ,I be an ideal of C.Then I is cosprime ideal if and only if I is cosemiprime ideal.

Proof :its follow directly from theorem (3.7)and theorem (3.8).

4. Conclusion

1. Every field is cosemiprime ring.
2. Every prime ring is cosemiprime ring.
3. Every cosemiprime ring is P.I.R,but the converse is not true.
4. Suppose I be cosprime ideal of cosemiprime ring C,then every zero divisor of C\I is a nilpotent element.
5. Let C be a cosemiprime ring ,I an ideal of C.Then I is a cosemiprime ring if and only if it is cosprimary ideal.
6. Suppose C be cosemiprimary ring,I be an ideal of C ,then I is coprimary ideal if and only if I is a cosemiprime ideal.

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6. References

[1] Lam,Tsit-Yuea;"Afirst Course in Non Commutative ring"s; New Yourk, Springer-Verlag 2001.
[2] Dummit, David, Foote, Richard M.; "Abstract Algebra *(3rd ed)*, 2003.
[3] Park,Younk Soo and Kim,Ju.Pil,(1992). Prime and Semiprime ideals in semigroups*, Kyungpook Math. Journal. 32:(* 3).pp: 629-633.
[4] Atiyah, Michael, Macdonald,lan.;" Introduction to Commutative Algebra; Addision waasley,1994
[5] Lam,Tsit-Yuea;" *Lectures onModules and rings*"; Graduate texts in Math., No.189, Berlian, New York, 1999, pp:207-286.
[6] Bourbaki, Nicolas; *Algebre (2nded.);* Berlin, NewYork; Spring-2012
[7] Nathan,Jacobsne; *Basic Algebra*;Mineola,New York 2009
[8] Atiyah, Michacl, Macdonald, Ian; *Introduction to cmmutative algebra; Addision, Sanfrancisco 1969*
[9] H. M. Abdullah, (2019).The integrals in the Abelian groups.*Global Scientific Journal*.4:(2).pp: 166-171.