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Numerical Solutions of FIDE Using Quadrature Method

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ABSTRACT

In this peeper, some numerical methods for solving first order nonlinear Fredholm Integro-Differential Equations [FIDEs] are presented. Moreover, in order to support the numerical results, some numerical experiments are given. Finally, we discuss the obtained numerical results.

1. Introduction

In this search we first present the most familiar formulas of numerical integration: the Quadrature rules (Trapezoidal rule, Bool's rule and Weddel's rule) and then we illustrate primarily the use of these rules for evaluating integrals and show how the linear FIDEs of first order is reduced to system of (n) equation in the (n) unknowns of the solution sample values $u(x_i)$, i=0,1,2,…n.

The procedure of the previous technique is called the Quadrature method.

In addition a computer program is written, examples with satisfactory results are given.

2. Quadrature Rules

A Quadrature rule is a generic name given to any numerical method for the approximate calculation of definite integral I[u] of the function u(x) over finite integral [a, b],

$$I[u]=\int_a^b u(x)\,dx, \quad a<b$$

Then numerical approximation to I[u] of the form

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$$I_n[u] = \sum_{j=0}^{n} w_j u(x_j)$$

is called an (n+1)-point Quadrature formula with approximation Error (or Quadrature error) $E_n[u]$,

i.e.

$$I[u] = I_n[u] + E_n[u]$$

where the points $x_j\,(j = 0,1,\ldots, n)$ are called the integration nodes which are lying in the interval [a,b] and $w_j\,(j = 0,1,\ldots, n)$ are constants which are called Quadrature weights [1,2].

The following are basic formulas of Quadrature approximation methods:

2.1 Trapezoidal Rule:

The familiar and simple trapezoidal rule is based on approximating u(x) on interval $(x_0 = a, b = x_1)$ by the straight line joining $(x_0, u(x_0))$ and $(x_1, u(x_1))$.

Hence using the area of trapezoid with height $\frac{1}{2}[u(x_0) + u(x_1)]$, gives [5]:

$$\int_{x_0}^{x_1} u(x)\,dx = \frac{h}{2}[u(x_0) + u(x_1)] - \frac{h^3}{12}u''(z) \quad \text{.... (1.1)}$$

where $z \in (x_0, x_1)$

when the length of interval [a,b] is not sufficiently small, the trapezoidal rule (1.1) is not of much use. For such an interval, we break it into a sum of integrals over small subinterval, and then apply (1.1) to each of these smaller integrals, we call it the composite trapezoidal rule.

Let $h = \frac{b-a}{n}, n \geq 1$ and $x_i = a + i * h(i = 0,1,\ldots\ldots, n)$; then it can be written as:

$$\int_a^b u(x)\,dx \cong \sum_{i=0}^{n} \frac{h}{2}[u(x_i) + u(x_{i+1})]$$

$$= \frac{h}{2}[u(a) + u(b) + 2\sum_{i=1}^{n-1} u(x_i)] \quad \text{.... (1.2)}$$

with global error [5].

$$E_n[u] = -\frac{(b-a)}{12} h^2 u''(z), z \in (a,b)$$

2.2 Bool's Rule:

The fourth formula approximates the function on (x_i, x_{i+4}) by a curve that passes through five points $(x_i, u(x_i))$, $(x_{i+1}, u(x_{i+1}))$, $(x_{i+2}, u(x_{i+2}))$, $(x_{i+3}, u(x_{i+3}))$ and $(x_{i+4}, u(x_{i+4}))$ which results in the composite Bool's rule.

$$\int_{x_0}^{x_n} u(x)\,dx = \frac{2h}{45}[7u(x_0) + 32u(x_1) + 12u(x_2) + 32u(x_3) + 14u(x_4) + \ldots\ldots$$
$$+ 12u(x_{2n-2}) + 32u(x_{2n-1}) + 7u(x_{2n})] \quad \ldots. (1.3)$$

with error $E_n[u]$ of order $O(h^7)$.

2.3 Weddel's Rule:

The fifth formula approximates the function on (x_i, x_{i+6}) by a curve that passes through seven points $(x_i, u(x_i))$, $(x_{i+1}, u(x_{i+1}))$, $(x_{i+2}, u(x_{i+2}))$, $(x_{i+3}, u(x_{i+3}))$, $(x_{i+4}, u(x_{i+4}))$, $(x_{i+5}, u(x_{i+5}))$ and $(x_{i+6}, u(x_{i+6}))$ which results in the composite Weddel's rule.

$$\int_{x_0}^{x_n} u(x)\,dx \cong \frac{3h}{10}[u(x_0) + 5u(x_1) + u(x_2) + 6u(x_3) + u(x_4) + 5u(x_5) + 2u(x_6) +$$
$$5u(x_7) + u(x_8) + \ldots\ldots + 6u(x_{n-3}) + u(x_{n-2}) + 5u(x_{n-1}) + u(x_n)] \quad \ldots. (1.4)$$

with error $E_n[u]$ of order $O(h^7)$.

For further details about these rules see [3,4].

3. Numerical Solution of FIDE Using Quadrature Method

In this section, we use Quadrature methods to find the numerical solution of 1^{st} order linear FIDE, in the form:

$$u'(x) + p(x)u(x) = f(x) + \int_a^b k(x,t)u(t)\,dt,\ x \in I = [a,b] \quad \text{.... (1.5)}$$

with the initial condition $u(a) = u_0$, where the functions f and p are assumed to be continuous on I and k denotes given continuous functions.

Are the interval [a, b] is divided in to n equal subintervals, where h=(b-a)/n, $y_0 = a, y_n = b$ and $y_j = a + j * h, j = 0,1,\ldots, n.$ we set $x_i = y_j, i = 0,1,\ldots, n, u'(x_i) = u_i', p(x_i) = p_i, f(x_i) = f_i,$ $u(x_i) = u_i$ and $k(x_i, y_j) = k_{ij}$.

3.1 Trapezoidal Rule:

Trapezoidal rule is used with n-subinterval to approximate the integral in equation (1.5), hence

$$u_i' + p_i u_i = f_i + \frac{h}{2}[k_{i0}u_0 + 2k_{i1}u_1 + \ldots + 2k_{ii-1}u_{i-1} + k_{ii}u_i] \quad \text{.... (1.6)}$$

since

$$u_i'(x) = \frac{u_i(x) - u_{i-1}(x)}{h} \quad \text{.... (1.7)}$$

Substituting equation (1.7) into equation (1.6) we have

$$(1 + hp_i - \frac{h^2}{2}k_{ii})u_i - u_{i-1} = hf_i + \frac{h^2}{2}[k_{i0}u_0 + 2k_{i1}u_1 + \ldots + 2k_{ii-1}u_{i-1}] \quad \text{.... (1.8)}$$

which are (n+1) equation in u_i that represents the approximate solution to equation (1.5) at $x = a + i * h, (i = 0,1,\ldots, n).$

$$\begin{bmatrix} 1 + hp_1 - \frac{h^2}{2}k_{11} & 0 & 0 & 0 & \cdots & 0 \\ -1 - h^2 k_{21} & 1 + hp_2 - \frac{h^2}{2}k_{22} & 0 & 0 & \cdots & 0 \\ -h^2 k_{31} & -1 - h^2 k_{32} & 1 + hp_3 - \frac{h^2}{2}k_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} hf_1 + \frac{h^2}{2}k_{10}u_0 + u_0 \\ hf_2 + \frac{h^2}{2}k_{20}u_0 \\ hf_3 + \frac{h^2}{2}k_{30}u_0 \\ \vdots \end{bmatrix}$$

The Algorithm (AQ1)

The numerical solution of (1^{st} order FIDEs), by using Quadrature methods (Trapezoidal rule), is obtained as follows:

Step 1: Put h=(b-a)/n, $n \in N$

Step 2: Set $u_0 = u(a)$ (which is the initial condition) is given.

Step 3: Compute u_i' by using $u_i' = \dfrac{u_i - u_{i-1}}{h}$

Step 4: Use step –1, -2 and –3 in equation (1.6) to find $u_i, (i = 1,2,\ldots\ldots, n)$ we get

$$(1 + hp_i - \frac{h^2}{2} k_{ii})u_i - u_{i-1} = hf_i + \frac{h^2}{2}[k_{i0}u_0 + 2k_{i1}u_1 + \ldots\ldots + 2k_{ii-1}u_{i-1}]$$

3.2 Bool's Rule:

Also Bool's rule can be used to approximate equation (1.5), which gives.

$$u_i' + p_i u_i = f_i + \frac{2h}{45}[7k_{i0}u_0 + 32k_{i1}u_1 + 12k_{i2}u_2 + 32k_{i3}u_3 + 14k_{i4}u_4 + 32k_{i5}u_5 + \ldots\ldots + 12k_{ii-2}u_{i-2} + 32k_{ii-1}u_{i-1} + 7k_{ii}u_i] \quad \ldots (1.9)$$

Substituting equation (1.7) in equation (1.9) gives.

$$(1 + hp_i - \frac{2h^2}{45} k_{ii})u_i - u_{i-1} = hf_i + \frac{2h^2}{45}[7k_{i0}u_0 + 32k_{i1}u_1 + 12k_{i2}u_2 + 32k_{i3}u_3 + 14k_{i4}u_4 + \ldots\ldots + 12k_{ii-2}u_{i-2} + 32k_{ii-1}u_{i-1}] \quad \ldots (1.10)$$

which are (n+1) equations in f_i that represents the approximate solution to equation (1.5) at $x=x_i=a+ih$, i=0,1,2,…n.

$$\begin{bmatrix} 1+hp_1-\frac{h^2}{2}k_{11} & 0 & 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ -1-\frac{4h^2}{3}k_{21} & 1+hp_2-\frac{h^2}{3}k_{22} & 0 & 0 & 0 & 0 & \cdots\cdots & 0 \\ -\frac{9h^2}{8}k_{31} & -1-\frac{9h^2}{8}k_{32} & 1+hp_3-\frac{3h^2}{8}k_{33} & 0 & 0 & 0 & \cdots\cdots & 0 \\ -\frac{64h^2}{45}k_{41} & -\frac{8h^2}{15}k_{42} & -1-\frac{64h^2}{45}k_{43} & 1+hp_4-\frac{14h^2}{45}k_{44} & 0 & 0 & \cdots\cdots & 0 \\ -\frac{73h^2}{90}k_{51} & -\frac{64h^2}{45}k_{52} & -\frac{8h^2}{15}k_{53} & -1-\frac{64h^2}{45}k_{54} & 1+hp_5-\frac{14h^2}{45}k_{55} & 0 & \cdots\cdots & 0 \\ -\frac{4h^2}{3}k_{61} & -\frac{29h^2}{45}k_{62} & -\frac{64h^2}{45}k_{63} & -\frac{8h^2}{15}k_{64} & -1-\frac{64h^2}{45}k_{65} & 1+hp_6-\frac{14h^2}{45}k_{66} & 0\ \cdots\cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ \vdots \end{bmatrix} = \begin{bmatrix} hf_1+\frac{h^2}{2}k_{10}u_0+u_0 \\ hf_2+\frac{h^2}{3}k_{20}u_0 \\ hf_3+\frac{3h^2}{8}k_{30}u_0 \\ hf_4+\frac{14h^2}{45}k_{40}u_0 \\ hf_5+\frac{h^2}{2}k_{50}u_0 \\ hf_6+\frac{h^2}{3}k_{60}u_0 \\ \vdots \end{bmatrix}$$

The Algorithm (AQ2)

The numerical solution of (1^{st} order FIDEs), by using Quadrature methods (Bool's rule), is obtained as follows:

Step 1: Put h=(b-a)/n, $n\in N$

Step 2: Set $u_0 = u(a)$ (which is the initial condition) is given.

Step 3: Compute u_i' by using $u_i' = \dfrac{u_i - u_{i-1}}{h}$

Step 4: Using step –1, -2 and –3 in equation (1.9) to find $u_i, (i = 1,2,\ldots\ldots, n)$ we get

$$(1+hp_i-\frac{2h^2}{45}k_{ii})u_i - u_{i-1} = hf_i + \frac{2h^2}{45}[7k_{i0}u_0 + 32k_{i1}u_1 + 12k_{i2}u_2 + 32k_{i3}u_3 + 14k_{i4}u_4 + \ldots\ldots + 12k_{ii-2}u_{i-2} + 32k_{ii-1}u_{i-1}]$$

3.3 Weddel's Rule:

Finally, Weddel's rule is used to approximate equation (1.5), which gives.

$$u_i' + p_i u_i = f_i + \frac{3h}{10}[k_{i0}u_0 + 5k_{i1}u_1 + k_{i2}u_2 + 6k_{i3}u_3 + k_{i4}u_4 + 5k_{i5}u_5 + 2k_{i6}u_6 + \ldots . + 6k_{ii-3}u_{i-3} + k_{ii-2}u_{i-2} + 5k_{ii-1}u_{i-1} + k_{ii}u_i]$$

.... (1.11)

Substituting equation (1.7) in equation (1.11) results in

$$(1+hp_i-\frac{3h^2}{10}k_{ii})u_i-u_{i-1}=hf_i+\frac{3h^2}{10}[k_{i0}u_0+5k_{i1}u_1+k_{i2}u_2+6k_{i3}u_3+k_{i4}u_4+5k_{i5}u_5+ \\ 2k_{i6}u_6+\ldots..+6k_{ii-3}u_{i-3}+k_{ii-2}u_{i-2}+5k_{ii-1}u_{i-1}]$$

.... (1.12)

Equations (1.12) is the approximate solution to equation (1.5) at x=x_i =a+ih, i=0,1,2,…n.

$$\begin{bmatrix}
1+hp_1-\frac{h^2}{2}k_{11} & 0 & 0 & 0 & 0 & 0 & & \cdots & 0 \\
-1-\frac{4h^2}{3}k_{21} & 1+hp_2-\frac{h^2}{3}k_{22} & 0 & 0 & 0 & 0 & & \cdots & 0 \\
-\frac{9h^2}{8}k_{31} & -1-\frac{9h^2}{8}k_{32} & 1+hp_3-\frac{3h^2}{8}k_{33} & 0 & 0 & 0 & & \cdots & 0 \\
-\frac{64h^2}{45}k_{41} & -\frac{8h^2}{15}k_{42} & -1-\frac{64h^2}{45}k_{43} & 1+hp_4-\frac{14h^2}{45}k_{44} & 0 & 0 & & \cdots & 0 \\
-\frac{73h^2}{90}k_{51} & -\frac{64h^2}{45}k_{52} & -\frac{8h^2}{15}k_{53} & -1-\frac{64h^2}{45}k_{54} & 1+hp_5-\frac{14h^2}{45}k_{55} & 0 & & \cdots & 0 \\
-\frac{3h^2}{2}k_{61} & -\frac{3h^2}{10}k_{62} & -\frac{9h^2}{5}k_{63} & -\frac{3h^2}{10}k_{64} & -1-\frac{3h^2}{2}k_{65} & 1+hp_6-\frac{3h^2}{10}k_{66} & 0 & \cdots & 0 \\
-\frac{4h^2}{5}k_{71} & -\frac{3h^2}{2}k_{72} & -\frac{3h^2}{10}k_{73} & -\frac{9h^2}{5}k_{74} & -\frac{3h^2}{10}k_{75} & -1-\frac{3}{2}k_{76} & 1+hp_7-\frac{3h^2}{10}k_{77} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & & &
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ \vdots \end{bmatrix}
=
\begin{bmatrix}
hf_1+\frac{h^2}{2}k_{10}u_0+u_0 \\
hf_2+\frac{h^2}{3}k_{20}u_0 \\
hf_3+\frac{3h^2}{8}k_{30}u_0 \\
hf_4+\frac{14h^2}{45}k_{40}u_0 \\
hf_5+\frac{h^2}{2}k_{50}u_0 \\
hf_6+\frac{3h^2}{10}k_{60}u_0 \\
hf_7+\frac{h^2}{2}k_{70}u_0 \\
\vdots
\end{bmatrix}$$

4. The Algorithm (AQ3)

The numerical solution of (1^{st} order FIDEs), by using Quadrature methods (Weddel's rule), is obtained as follows:

Step 1: Put h=(b-a)/n, $n\in N$

Step 2: Set $u_0=u(a)$ (which is the initial condition) is given.

Step 3: Compute u_i' by using $u_i'=\frac{u_i-u_{i-1}}{h}$

Step 4: Using step –1, -2 and –3 in equation (1.11) to find $u_i,(i=1,2,\ldots..,\ n)$ yields

$$(1+hp_i-\frac{3h^2}{10}k_{ii})u_i-u_{i-1}=hf_i+\frac{3h^2}{10}[k_{i0}u_0+5k_{i1}u_1+k_{i2}u_2+6k_{i3}u_3+k_{i4}u_4+5k_{i5}u_5+$$

$$2k_{i6}u_6+5k_{i7}u_7+\ldots\ldots+6k_{i-3}u_{i-3}+k_{ii-2}u_{i-2}+5k_{ii-1}u_{i-1}]$$

4. Numerical Examples:

Example (1):

Consider the following FIDE:

$$u'(x) + 3u(x) = 10x + 15x^2 - \frac{35}{12}x^4 + \int_0^x (x+y)u(y)\,dy \qquad 0 \le x \le 1$$

The exact solution is $u(x) = 5x^2$, Take n=10 h=0.1 and x_i =a+ih, i=0, 1, …., n.

Table (1-1) illustrates the comparison between the exact and numerical solution depending on the least square error and running time.

X	EXACT	TRAP.	BOOL'S	WEDDEL'S
0.0	0.000000000	0.000000000	0.000000000	0.000000000
0.1	0.050000000	0.088507185	0.088507185	0.088507185
0.2	0.200000000	0.268340554	0.268270955	0.268270955
0.3	0.450000000	0.541768532	0.541564832	0.541564832
0.4	0.800000000	0.910607117	0.910186884	0.910186884
0.5	1.250000000	1.376306556	1.375615385	1.375615385
0.6	1.800000000	1.940026422	1.938900151	1.938900127
0.7	2.450000000	2.602699740	2.601015239	2.601069785
0.8	3.200000000	3.365087178	3.362706645	3.362749843
0.9	4.050000000	4.227822600	4.224658368	4.224626421
1.0	5.000000000	5.191451322	5.187251817	5.187227918
L.S.E.		0.181214850	0.176589315	0.176599737
R.T.		0.330000000	0.330000000	0.330000000

Table (1-1)

Table (1-1.a) gives the least square error and running time for different values of n.

QUADRATURE RULES	N=15		N=20		N=30	
	L.S.E	R.T.	L.S.E.	R.T.	L.S.E.	R.T.
Trapezoidal	0.118494708	0.610000000	0.088096716	1.040000000	0.058257197	2.090000000
Bool's	0.116592127	0.650000000	0.087068918	1.040000000	0.057819264	2.080000000
Weddel's	0.116600535	0.600000000	0.087072843	1.040000000	0.057820112	2.090000000

Table (1-1.a)

Example (2):

Consider the following Fredholm integro-differential equation:

$$u'(x) = 1 - \int_0^x u(t)\,dt \qquad 0 \le x \le 1$$

Table (1-2) presents results from a computer program that solves this problem for which the analytical solution is $u(x) = \sin x$ over the interval x=0 to x=1 with h=0.1.

X	EXACT	TRAP.	BOOL'S	WEDDEL'S
0.0	0.00000000	0.000000000	0.000000000	0.000000000
0.1	0.09983341	0.099502487	0.099502487	0.099502487
0.2	0.19866933	0.197519863	0.197517396	0.197517396

0.3	0.29552020	0.293084218	0.293076884	0.293076884
0.4	0.38941834	0.385256868	0.385241577	0.385241577
0.5	0.47942553	0.473137546	0.473111395	0.473111395
0.6	0.56464247	0.555873170	0.555831032	0.555831033
0.7	0.64421768	0.632666098	0.632603221	0.632604061
0.8	0.71735609	0.702781787	0.702693078	0.702693899
0.9	0.78332690	0.765555788	0.765436742	0.765436709
1.0	0.84147098	0.820400009	0.820244323	0.820244275
L.S.E.		0.001246770	0.001262890	0.001262850
R.T.		0.330000000	0.330000000	0.330000000

Table (1-2)

Table (1-2.a) gives the least square error and running time for different values of n.

QUADRATURE RULES	**N=15**		**N=20**		**N=30**	
	L.S.E	**R.T.**	**L.S.E.**	**R.T.**	**L.S.E.**	**R.T.**
Trapezoidal	7.7138e-004	0.600000000	5.0025e-004	1.150000000	3.5617e-004	1.92000000
Bool's	7.7749e-004	0.600000000	5.0279e-004	1.150000000	3.5745e-004	2.03000000
Weddel's	7.7747e-004	0.600000000	5.0279e-004	1.210000000	3.5745e-004	2.03000000

Table (1-2.a)

Example (3):

Consider the following Fredholm integro-differential equation:

$$u'(x) + xu(x) = e^{x} + x + \int_{0}^{x} xu(y)\,dy \qquad 0 \le x \le 1$$

The exact solution is $u(x) = e^{x}$

Take n=10 h=0.1 and x_i =a+ih, i=0, 1, …., n.

Table (1-3) presents a comparison between the exact and numerical solution of the five types which depend on least square error and running time.

X	**EXACT**	**TRAP.**	**BOOL'S**	**WEDDEL'S**
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	1.105170918	1.110467649	1.110467649	1.110467649
0.2	1.221402758	1.232413013	1.232409259	1.232409259
0.3	1.349858807	1.366968921	1.366956481	1.366956481
0.4	1.491824697	1.515405384	1.515377046	1.515377046
0.5	1.648721270	1.679144426	1.679094946	1.679094946
0.6	1.822118800	1.859775499	1.859688188	1.859688182
0.7	2.013752707	2.059071724	2.058931747	2.058937685
0.8	2.225540928	2.279007240	2.278796832	2.278802425
0.9	2.459603111	2.521775989	2.521481811	2.521479523
1.0	2.718281828	2.789812301	2.789401965	2.789399952
L.S.E.		0.017236216	0.017094645	0.017095207
R.T.		0.330000000	0.280000000	0.320000000

Table (1-3)

Table (1-3.a) gives the least square error and running time for different values of n.

UADRATURE RULES	N=15		N=20		N=30	
	L.S.E	R.T.	L.S.E.	R.T.	L.S.E.	R.T.
Trapezoidal	0.010888400	0.600000000	0.007945296	0.990000000	0.005151653	1.970000000
Bool's	0.010832903	0.600000000	0.007916065	0.989999999	0.005139523	1.980000000
Weddel's	0.010833295	0.600000000	0.007916245	0.980000000	0.005139564	2.030000000

Table (1-3.a)

5. Discussion

The approximate solution of linear Fredholm integro-differential equation is given using the Quadrature methods. A computer program was written and several examples were solved using these methods.

Relying on our work the following notes are drawn:

1. The number of subintervals n is restricted to be even Bool's rule and a multiple of six for Weddel's rule.
2. Through the solution of linear Fredholm integro-differential equations of the first order, we see that Bool's rule gives the best results when it is compared with other Quadrature rules. See examples (1.1), (1.2) and (1.3).

6. References

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